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Representations of group planar algebras

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Abstract

We explicitly find out the irreducible representations of the planar algebra corresponding to the subfactor arising from the action of a finite group. We also answer the question posed by Vaughan Jones on the radius of convergence of the dimension of a representation in the affirmative for the case of group planar algebras.

Keywords: Subfactor; Planar algebra; Representation; Annular category; Quantum double

0. Introduction

In [Jon1], Jones introduced the notion of index for type II_1 subfactors and also examined the canonical tower,

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

of basic construction of $N \subset M$. For a finite index inclusion of type II_1 factors, $N \subset M$, the grid of finite dimensional algebras of relative commutants,

$$\begin{array}{ccccccccc} N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & \cdots \end{array}$$

known as the standard invariant, became an important invariant for $N \subset M$ (see [GHJ,JS,Pop1,Pop2]).

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Popa [Pop2] studied the question of which families $\{A_{ij} : -1 \leqslant i \leqslant j < \infty\}$ of finite-dimensional C^*-algebras could arise as the tower of relative commutants of an extremal finite-index subfactor, i.e., when does there exist such a subfactor $M_{-1} \subset M_0$ such that $A_{ij} = M_i' \cap M_j$; and he obtained a beautiful algebraic axiomatization of such families, which he called *λ-lattices*. Subsequently, Jones used this characterisation of λ-lattices to obtain an algebraic and geometric reformulation of the standard invariant, which he called *Planar Algebras* (see [Jon2]). Jones then introduced the notion of '*modules over a planar algebra*' in [Jon3] where he explicitly found the irreducible modules over the Temperley–Lieb planar algebras for index greater than 4. This notion became a powerful tool to construct subfactors of index less than 4, namely, the subfactors with principal graph, E_6 and E_8.

Probably the most accessible planar algebra, after the Temperley–Lieb planar algebras, is the group planar algebra. We explicitly find the irreducible modules over the group planar algebra. Jones [Jon3] asked whether the dimension of any irreducible module over a planar algebra has radius of convergence at least as big as $\frac{1}{\delta^2}$ where δ is the modulus of the planar algebra. We verify the truth of this assertion in the case of group planar algebras.

In Section 1.1, we rephrase the definition of annular category over planar algebras as in [Jon3] but in a slightly different language. We first define this category for a particular presentation of the planar algebra and later show that it is independent of the presentation (see Proposition 1.2). We extensively make use of the fact that in any annular tangle we can, by isotopy, 'drag all internal discs around the distinguished internal disc, and place inside a single box' and finally get a much simpler annular tangle.

We consider the representations of a planar algebra in Section 1.2, or equivalently (see [Jon3]), modules over planar algebras. As in [Jon3], we describe an explicit algorithm to find the irreducible representations.

Section 2.1 is devoted to finding a basis of the space of morphisms of the annular category of the group planar algebra. It becomes clear that every labelled annular tangle comes from a regular tangle, i.e., a (non-annular) tangle which may or may not have labelled internal discs. We define a set of relations over regular tangles and establish a correpondence between annular tangles and equivalence classes of regular ones. This correspondence is given by a set of moves. (Although we restrict here to the group planar algebras, a careful inspection of the proof shows that this correspondence can be generalized to arbitrary planar algebras.) A crucial move here is one coming from a 'rotation of the internal disc by 360°.

In Section 2.2, we identify the algebraic structures of the space of morphisms of the annular category in the group case. The relation corresponding to the twist of the internal disc by 360° plays an important role in identifying the algebra, $(AP)_1^1$ as a non-trivial quotient of the quantum double of the group. We make use of the explicit basis of the space of morphisms to show the absence of irreducible representations with weight greater than or equal to 2. We describe all the irreducible representations of weight 0 and 1. Finally, we find the dimensions of these irreducible representations, and verify—at least for the case of group planar algebras—the truth of a conjecture of

[Jon3], by checking that the dimension of every irreducible representation has a radius of convergence of $\frac{1}{\delta^2}$ where δ^2 is the cardinality of the group.

1. Annular category over a planar algebra and representations

This section is devoted to recalling from [Jon3], as well as setting up notations, the definitions and basic facts concerning the *annular category over a planar algebra* (in Section 1.1), and representations of a planar algebra, referred to as 'modules over the planar algebra' in [Jon3] (in Section 1.2).

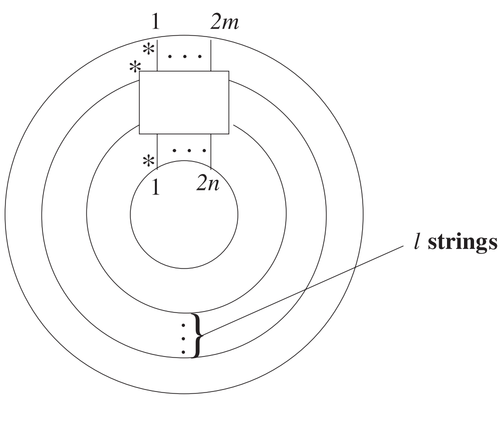
1.1. Annular category over a planar algebra

Let P be a planar algebra (as defined in [Jon2]) with Φ as the presenting map and $L = \{L_k\}_{k\in\mathbb{N}}$ as the labelling set or the set of generators. Let Col be the set $\{0^+, 0^-, 1, 2, 3, \ldots\}$. Here, we will mostly deal with *annular tangles* and will not restrict our attention only to their actions at the level of planar algebra. We first define *annular tangles* and set some notations.

Definition 1.1. An annular (m,n)-tangle $(m, n \in Col)$ is simply an m-tangle with a distinguished internal n-disc. If m (resp. n) is 0^+, then the region adjacent to the external (resp. internal) boundary is unshaded and for 0^-, it is shaded. An L-labelled annular (m,n)-tangle is an annular (m,n)-tangle in which each of the internal discs, with the exception of the distinguished one, has been L-labelled.

Note that each annular tangle is an equivalence class, where the equivalence is given by ambient isotopy in the annulus.

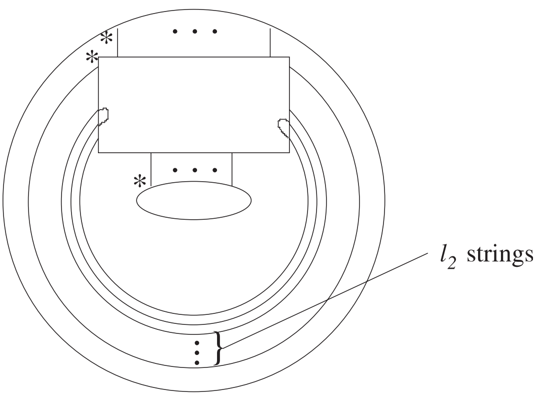
Let $(AnnL)_n^m$ be the set of all L-labelled annular (m,n)-tangles and $(FAL)_n^m$ be the vector space with basis $(AnnL)_n^m$ over $\mathbb{C}$. We can (following [Jon3]) compose $A \in (AnnL)_j^i$ and $B \in (AnnL)_k^j$ and form the composition $A \cdot B \in (AnnL)_k^i$. We linearly extend this composition to obtain a bilinear map $(FAL)_j^i \times (FAL)_k^j \mapsto (FAL)_k^i$. Now, $(FAL)_n^m$ is a huge vector space and we need to 'mod out' the relations given by the kernel of the presenting map Φ. To make this precise, we shall define some special tangles. For $m, n \in \mathbb{N}$, $l \in \mathbb{N}$, let $\Psi_{m,n}^l$ be the following tangle:



If $m = 0^{\pm}$ (resp. $n = 0^{\pm}$), then $\Psi^{l}_{m,n}$ will exactly be same as the above tangle except that there will be no strings connecting the top (resp. bottom) of the box with the external (resp. internal) boundary of the annulus; if $m = 0^{-}$, then we take the topmost end point of a string on the left of the box as its first point; also it is apparent that $l \in \mathbb{N}_{\text{even}}$ unless we have the cases, $m = 0^{-} \neq n$ or $m \neq 0^{-} = n$, in which $l \in \mathbb{N}_{\text{odd}}$.

It is easily seen that each $\Psi^{l}_{m,n}$ induces a linear map from $\mathcal{P}_{l+m+n}(L)$ (the 'space of $(l+m+n)$-boxes' in the universal planar algebra) into $(FAL)^m_n$, carrying the set $\mathcal{T}_{l+m+n}(L)$ of L-labelled $(l+m+n)$-tangles into $(AnnL)^m_n$; we will denote this map by $\psi^{l}_{m,n}$. More explicitly, this is done by 'plugging the input from $\mathcal{P}_{l+m+n}(L)$ into the internal box of colour $(l+m+n)$ in $\Psi^{l}_{m,n}$'. It is also worth noticing that, using isotopy, any L-labelled annular (m,n)-tangle, with at least one string in its interior, can be expressed as the $\psi^{l}_{m,n}$-image of some L-labelled $(l+m+n)$-tangle, for some $l \in 2\mathbb{N}$. Thus by linearity of $\psi^{l}_{m,n}$, any element of $(FAL)^m_n$ is a $\psi^{l}_{m,n}$-image of some element of $\mathcal{P}_{l+m+n}(L)$ for some $l \in \mathbb{N}$. The reader should note that an element of $(FAL)^m_n$ can be expressed as $\psi^{l}_{m,n}(X)$ for $X \in \mathcal{P}_{l+m+n}(L)$, $l \in \mathbb{N}$ in more than one way—in fact, in infinitely many ways.

Let $\mathcal{W}(L)^m_n = \left(\bigcup_{l \in 2\mathbb{N}} \psi^{l}_{m,n}(kernel(\Phi) \cap \mathcal{P}_{l+m+n}(L))\right)$. It is then a fact that $\mathcal{W}(L)^m_n$ is a vector subspace of $(FAL)^m_n$. Let $X_i \in kernel(\Phi) \cap \mathcal{P}_{l_i+m+n}(L)$, for $i = 1, 2$ and let $l_1 \geqslant l_2$. Now, $\psi^{l_2}_{m,n}$ can be isotoped to the following picture:



We keep wiggling the strings around the internal disc until the total number of string around the disc increases from l_2 to exactly l_1. Thus we express $\psi^{l_2}_{m,n}(X_2)$ as $\psi^{l_1}_{m,n}(X)$, where X is same as X_2 with few more extra caps on its boundary. Since $X_2 \in kernel(\Phi)$ and X is given by an annular tangle acting on X_2, therefore $X \in kernel(\Phi)$. Thus $\psi^{l_1}_{m,n}(X_1) + \psi^{l_2}_{m,n}(X_2) = \psi^{l_1}_{m,n}(X_1) + \psi^{l_1}_{m,n}(X) = \psi^{l_1}_{m,n}(X_1 + X) \in \mathcal{W}(L)^m_n$.

From the definition of $\mathcal{W}(L)^m_n$, one can verify that the image of $\mathcal{W}(L)^i_j \times (APL)^j_k$ and $(APL)^i_j \times \mathcal{W}(L)^j_k$, under the composition map, are subsets of $\mathcal{W}(L)^i_k$. In particular, we see that composition induces an algebra structure in $(FAL)^m_m$ and that $\mathcal{W}(L)^m_m$ is an ideal in it.

The space $(APL)^m_n$ is defined as the quotient of $(FAL)^m_n$ by $\mathcal{W}(L)^m_n$. (Thus, for instance, $(APL)^m_m$ is an algebra.) We shall denote the image of an element $B \in (FAL)^m_n$ in $(APL)^m_n$ by $[B]$. It can be checked that composition factors through this quotient; i.e., the rule

$$(FAL)^i_j \times (FAL)^j_k \ni (A, B) \;\mapsto\; A \cdot B \;\in (FAL)^i_k$$

is seen to descend to a well defined and bilinear map from $(APL)^i_j \times (APL)^j_k$ into $(APL)^i_k$.

Proposition 1.2. *For any two presentations of a planar algebra P, say on label sets L and L', $(APL)^m_n$ is isomorphic to $(APL')^m_n$ for each $m, n \in Col$, via an isomorphism that respects the composition.*

Proof. It is enough to show isomorphism for the case $L' = P$. Let $\alpha : (AnnL)^m_n \to (AnnP)^m_n$ be defined by as follows:

Fix $A \in (AnnL)^m_n$; let $\{l_i\}_{i=1}^r$ be the labels on the internals discs of A other than the distinguished one. Let $x_i = \Phi(l_i)$ for each $i = 1, 2, \ldots, r$. Set $\alpha(A)$ as the P-labelled annular (m,n)-tangle formed by replacing the label l_i by x_i for $i = 1, 2, \ldots, r$.

Clearly, α is a well-defined map preserving composition; α is extended linearly to $\beta : (FAL)^m_n \to (FAP)^m_n$ which again preserves composition. Let $\tilde{\beta}$ be the map β composed with the quotient map of $(FAP)^m_n$ over $\mathcal{W}(P)^m_n$. A moment's thought will establish that $\tilde{\beta}$ is surjective.

To prove $kernel(\tilde{\beta}) = \mathcal{W}(L)^m_n$, let $B \in kernel(\tilde{\beta})$. This implies $\beta(B)$ can be expressed as $\psi^l_{m,n}(X)$ for some $X \in \mathcal{P}_{l+m+n}(P)$, $l \in 2\mathbb{N}$. Let $B = \sum_{i \in I} \lambda_i A_i$ where A_i's are distinct elements of $(AnnL)^m_n$ and $\lambda_i \in \mathbb{C}$; also let $X = \sum_{j \in J} \mu_j T_j$ where T_j's are in $\mathcal{T}_{l+m+n}(P)$ and $\mu_j \in \mathbb{C}$. From the equation $\beta(B) = \psi^l_{m,n}(X)$, we can conclude the following facts:

(a) J can be partitioned into disjoint subsets $\{J_k\}_{k \in K}$ such that $\psi^l_{m,n}(T_{j'})$ and $\psi^l_{m,n}(T_{j''})$ are the same if and only if $j', j'' \in J_k$ for some $k \in K$,
(b) for each $i \in I$, there exists unique $k_i \in K$ such that $\beta(A_i) = \psi^l_{m,n}(T_j)$ for all $j \in J_{k_i}$,
(c) $\lambda_i = \sum_{j \in J_{k_i}} \mu_j$ for each $i \in I$,
(d) $\sum_{j \in J_k} \mu_j = 0$ for $k \in K \setminus \{k_i : i \in I\}$.

Note that for each $k \in K$ and $j', j'' \in J_k$, the equation, $\psi^l_{m,n}(T_{j'}) = \psi^l_{m,n}(T_{j''})$, will induce a bijection between the sets of P-labelled internal discs of $T_{j'}$ and $T_{j''}$; in fact, the sets are equal. For each $k \in K$, fix a representative, say T_k, of $\{T_j\}_{j \in J_k}$. For each label, y with colour r of T_k, fix $C_y \in \mathcal{P}_r(L)$ so that $\Phi(C_y) = y$; and in the case $k = k_i$, for some $i \in I$, we already have a label l corresponding to y given by the equation $\beta(A_i) = T_k$—set $C_y = l$ in this case. Now, for each $j \in J$, form the tangle, $S_j \in \mathcal{P}_{l+m+n}$ by replacing the label y in T_j by C_y. Using (a), we can conclude that $\psi^l_{m,n}(S_{j'})$ and $\psi^l_{m,n}(S_{j''})$ are the same if and only if $j', j'' \in J_k$ for some

$k \in K$; this fact along with (b), (c) and (d) will imply $B = \psi^l_{m,n}(\sum_{j\in J}\mu_j S_j)$. Again, $\Phi(\sum_{j\in J}\mu_j S_j) = \sum_{j\in J}\mu_j \Phi(S_j) = \sum_{j\in J}\mu_j \Phi_P(T_j) = \Phi(X) = 0$ where Φ_P is the presenting map of P on itself. Thus, $B \in \mathcal{W}(L)^m_n$ proving $kernel(\tilde{\beta}) \subset \mathcal{W}(L)^m_n$. It is easy to see the reverse inclusion right from the definition of β.

Thus $\tilde{\beta}$ induces an isomorphism from the quotient $(APL)^m_n$ to $(APP)^m_n$ which preserves the composition. $\square$

So henceforth, we will write $(AP)^m_n$ instead of $(APL)^m_n$. For a C^* planar algebra, P, we have an obvious involution $(AnnL)^m_n \ni A \mapsto A^* \in (AnnL)^n_m$ where A^* is obtained by reflecting the unlabelled annular tangle A about its external boundary, the $*$'s for the boundaries of the discs remain in their reflected positions and if l is the label of a disc in A, then the corresponding disc in the reflected annular tangle will have l^* as its label. This map is extended conjugate linearly to define from $\{(FAP)^m_n\}_{m,n\in Col}$ to itself. Note that, for $X \in \mathcal{P}_{l+m+n}(L)$, $(\psi^l_{m,n}(X))^* = \psi^l_{n,m}(X')$ where X' is some rotation acting on X^*. So, if $X \in kernel(\Phi)$, then $X' \in kernel(\Phi)$ which implies that the involution, $*$ on $\{(FAL)^m_n\}_{m,n\in Col}$ is invariant on the subspace, $\{\mathcal{W}(L)^m_n\}_{m,n\in Col}$. Hence $*$ induces an involution on $\{(AP)^m_n\}_{m,n\in Col}$.

We regard (AP) $(= \{(AP)^m_n\}_{m,n\in Col})$ as a category (as in [Jon3]) with Col as the set of objects and $(AP)^m_n$ as the set of morphisms from n to m, for $m, n \in Col$. In fact, this category can be given a C^*-category structure (see [Jon3]).

Fix $k \in \mathbb{N}$. Let

$$\widehat{(AP)}_k = \left\{ A \in (AP)^k_k \ : \ \begin{array}{c} A \text{ is a linear combination of} \\ \text{elements of the form } B \cdot C \\ \text{where } B \in (AP)^k_n,\ C \in (AP)^n_k \\ \text{for } n \in Col \text{ such that } n < k \end{array} \right\}.$$

It is easy to see that $\widehat{(AP)}_k$ is an ideal in the algebra $(AP)^k_k$ (see [Jon3]. This $\widehat{(AP)}_k$ will play a vital role when we deal with irreducible representation of a planar algebra in the latter sections.

1.2. Representations of a planar algebra

Let *Vect* be the category of vector spaces.

Definition 1.3. A representation of a planar algebra, P is defined as a functor, F from the category (AP) to *Vect* such that the map $Mor_{(AP)}(n, m) = (AP)^m_n \ni a \mapsto F(a) \in \mathcal{L}(F(n), F(m))$ is a linear map, for all objects, $n, m \in Col$.

Remark 1.4. Every such representation F may be viewed as a graded vector space $\{F(n)\}_{n\in Col}$ with certain compatibility conditions, as in [Jon3] where representations are referred to as 'P-modules'. Clearly, $F(n)$ forms a module over the algebra $(AP)^n_n$ for each $n \in Col$.

Remark 1.5. The functor F, defined by $F(n) = P_n$ for all $n \in Col$ and $F(a)$ being the usual map from P_n to P_m for $a \in (AP)_n^m$, forms a representation and is called the 'trivial' representation.

Lemma 1.6. *For a representation, F, if $F(n) = \{0\}$ for some $n \in \mathbb{N}$, then $F(k) = \{0\}$, for all k in Col such that $k \leqslant n$.*

Proof. Let I_{n-1}^n be the inclusion tangle in $(AnnP)_{n-1}^n$. Note that we have the conditional expectation tangle $E_n^{n-1} \in (AnnP)_n^{n-1}$ for which $(E_n^{n-1} \circ I_{n-1}^n)$ is a non-zero multiple (namely, δ, the modulus of the planar algebra) of the identity tangle of $(AnnP)_{n-1}^{n-1}$. Thus, by functoriality of F, we get $F(n-1) = \{0\}$ and this process is repeated to show that $F(k) = \{0\}$, for all $k < n$. $\square$

Definition 1.7. The weight of a representation, F is the smallest k such that $F(k) \neq 0$ and is denoted by $wt(F)$. If $F(0^+)$ or $F(0^-)$ is non-zero, then set $wt(F) = 0$.

We will mostly be interested in spherical C^*-planar algebras (see [Jon2]). Let *Hil* be the category of finite dimensional Hilbert spaces.

Definition 1.8. A $*$-representation of a C^*-planar algebra P is a functor $F : (AP) \to Hil$ such that

(i) the map $Mor_{(AP)}(n, m) = (AP)_n^m \ni a \mapsto F(a) \in \mathcal{B}(F(n), F(m))$ is a linear map, for all objects, $n, m \in Col$,
(ii) $F(a^*) = F(a)^*$, for all $a \in Mor_{(AP)}$.

Remark 1.9. A $*$-representation corresponds to a Hilbert P-module in [Jon3].

Remark 1.10. Note that condition (ii) in Definition 1.8 says that:

$$\langle F(a)v, w\rangle = \langle v, F(a^*)w\rangle,$$

for all $n, m \in Col$, $a \in (AP)_n^m$, $v \in F(n)$, $w \in F(m)$ $n, m \in Col$.

We now restrict our attention to the class of spherical C^*-planar algebras P, and its $*$-representations. We say that two $*$-representations are *isomorphic* if there exists a natural isometric isomorphism between the corresponding functors; a $*$-representation G is defined as a *subrepresentation* of the $*$-representation F, if there exist a natural isometric transformation from G to F. The notions *direct sum*, *irreducibility* and *orthogonality* of subrepresentations are defined in the obvious manner (cf. [Jon3]). For example, the trivial representation is an irreducible $*$-representation (see [Jon3]). We list a few facts whose proof can be found in [Jon3]:

(i) F is irreducible iff $F(n)$ is irreducible as an $(AP)_n^n$-module for each $n \in Col$,

(ii) if W is an irreducible $(AP)_k^k$-submodule of $F(k)$ for some $k \in Col$, then W generates an irreducible subrepresentation of F,
(iii) orthogonal $(AP)_k^k$-submodules of $F(k)$, for some $k \in Col$, generate orthogonal subrepresentations of F,
(iv) if F and G are representations with F being irreducible and if $\theta : F(k) \to G(k)$ is a non-zero $(AP)_k^k$ homomorphism for some $k \in Col$, then θ extends to an injective homomorphism from F to G, that is, an injective natural transformation from F to G,
(v) fix $k \in \mathbb{N}$, and let $W_k = span\{F(a)v : a \in (AP)_n^k, v \in F(n), k > n \in Col\}$, then

$$W_k^{\perp} = \bigcap_{a \in \widehat{(AP)_k}} kernel(F(a)).$$

From (v), we can conclude that for a $*$-representation F with weight k, we have

$$F(k) = \bigcap_{a \in \widehat{(AP)_k}} kernel(F(a))$$

since W_k turns out to be zero and hence $F(k)$ forms a module over the quotient $\frac{(AP)_k^k}{\widehat{(AP)_k}}$. We denote this quotient algebra by $(LWP)_k$ (Lowest Weight algebra with weight k). By (i), if F is an irreducible $*$-representation with weight k, then $F(k)$ is an irreducible module over $(LWP)_k$. In order to find the irreducible $*$-representations of a C^*-planar algebra P, it suffices, as in [Jon3], to do the following:
(i) find the irreducible representations of $(LWP)_k$,
(ii) find which irreducible representation of $(LWP)_k$ gives rise to an irreducible $*$-representation of the planar algebra.

We will follow this method to list the irreducible $*$-representations of the group planar algebra in the next section.

Definition 1.11. The dimension of a representation F of a planar algebra is the formal power series

$$\frac{1}{2}\{\, dim(F(0^+)) + dim(F(0^-))\ \} \ + \ \sum_{k=1}^{\infty} dim(F(k))\ z^k,$$

denoted by $\Phi_F(z)$.

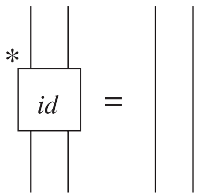
Clearly, this dimension is additive under direct sum of representations. We conclude this section by stating a question on dimensions of representations from [Jon3].

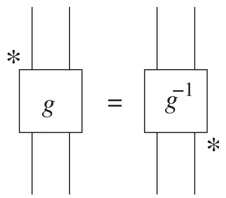
Question 1.12. *Is the radius of convergence of the dimension of an irreducible representation greater than or equal to δ^{-2}, where δ is the modulus of the planar algebra?*

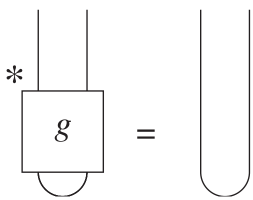
2. Annular category over the group planar algebra and representations

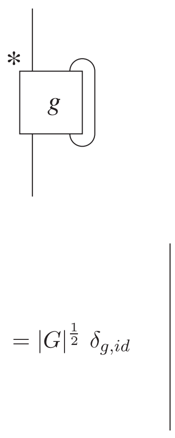
2.1. Annular category over the group planar algebra

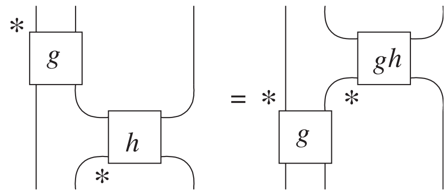
In this section, we will find an explicit basis of $(AP)_n^m$ where $P = P(G)$ denotes the group planar algebra for a finite group G. We consider the group planar algebra presentation for P, described in [Jon3,Lan], where the labelling set is $L = L_2 = G$ and the relations which generate the kernel of the presenting map, Φ are as follows:

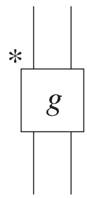
00. 

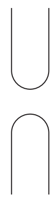
0. 

1. 

2. 

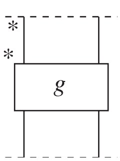
3. 

4. $\frac{1}{|G|}\sum_{g\in G}$ 

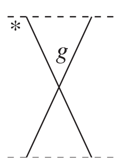
$= \frac{1}{|G|^{\frac{1}{2}}}$ 

Before going into details, we introduce some notation. Fix $g \in G$, $m \in \mathbb{N}$. Let $\alpha_m^g : \langle g\rangle \to Sym(G^m)$ be the left regular action, of the subgroup $\langle g\rangle$ of G generated by g, on the m-fold product of G: i.e., let $\alpha_m^g(t)\underline{g} = (tg_1, \ldots, tg_m)$, $\forall t \in \langle g\rangle$, $\underline{g} \in G^m$. Let S_m^g denote a subset of G^m such that (i) S_m^g meets each orbit of α_m^g in exactly one point and (ii) if $\underline{s} \in S_m^g$ and $s_1 \in \langle g\rangle$, then $s_1 = 1$. (Note that the cardinality of S_m^g is given by $\frac{|G|^m}{o(g)}$, where $o(g)$ denotes the order of g.) Therefore, for each $\underline{y} \in G^m$, there exists a unique element $\underline{s}_{\underline{y}}^g \in S_m^g$ such that $\langle g\rangle \cdot \underline{s}_{\underline{y}}^g = \langle g\rangle \cdot \underline{y}$, where we simply write $t \cdot \underline{h}$ for $\alpha_m^g(t)(\underline{h})$.

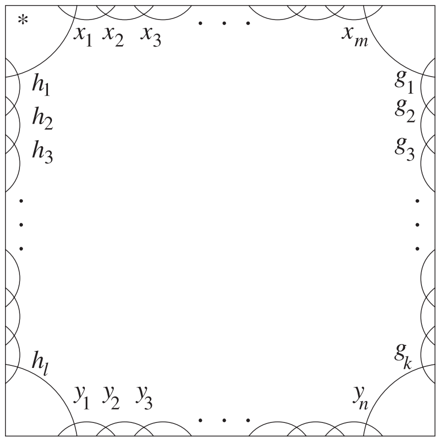
For convenience, we will shrink any g-labelled box, appearing inside a tangle, to a point and put the label g near the point in the shaded region which had the first point of the g-labelled box in its boundary; there is no loss of information in this re-writing. Thus,



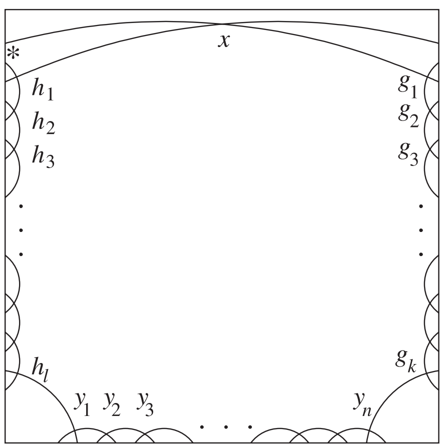
will be replaced by



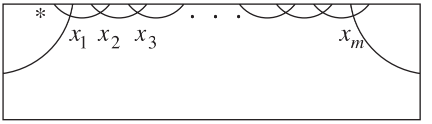
We will use $T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}}$ to denote the following element in $\mathcal{T}_{k+l+m+n}$:



where $\underline{x} \in G^m$, $\underline{y} \in G^n$, $\underline{g} \in G^k$, $\underline{h} \in G^l$ and $k, l, m, n \in \mathbb{N}$. (However, we will only use the case $k = l$ in this paper.) We will also use $T^{0^+,\underline{y}}_{\underline{g},\underline{h}}$ (resp. $T^{\underline{x},0^+}_{\underline{g},\underline{h}}$) to denote the above tangle with no boxes labelled $x_1, x_2, \cdots, x_m$ in the top (resp. $y_1, y_2, \cdots, y_n$ in the bottom) and with a straight horizontal string connecting the boxes labelled g_1 and h_1 (resp. g_k and h_l); $T^{0^+,0^+}_{\underline{g},\underline{h}}$ is defined in the obvious manner. Further, $T^{x^{\{0^-\}},\underline{y}}_{\underline{g},\underline{h}}$ will denote the following tangle:



where $x \in G$; again $T^{\underline{x},y^{\{0^-\}}}_{\underline{g},\underline{h}}$ and $T^{x^{\{0^-\}},y^{\{0^-\}}}_{\underline{g},\underline{h}}$ are defined in the obvious manner. Finally, $T^{\underline{x},0^-}$, $T^{0^-,\underline{y}}$, $T^{0^-,g,0^-}$ will denote the following tangles, respectively:



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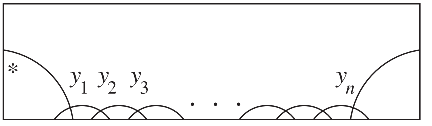

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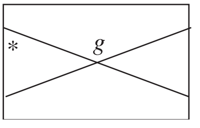
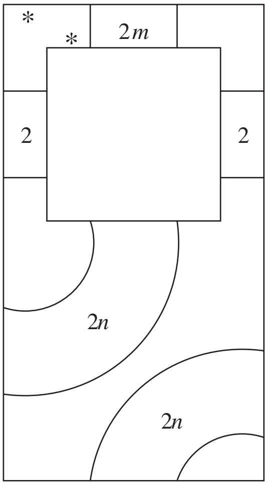


.

In the following discussion, we will extensively use the basis of $P_{k+l+m+n}$, described in [Lan], namely,

$$\{\Phi(T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}}) : \underline{x} \in G^m \text{ such that } x_1 = 1, \underline{y} \in G^n, \underline{g} \in G^k, \underline{h} \in G^l\}.$$

We also denote a band of k strings just by the two strings forming the boundary of the band between which we write k, for $k \geqslant 2$. For example, we shall need the annular tangle—which we shall denote by $(twist)^m_n$—given as follows (where the internal box has 'colour' $m+n+2$):



Note that $\psi^{2n+2}_{m,n}((twist)^m_n(T)) = \psi^2_{m,n}(T)$ for all $T \in \mathcal{T}_{2+m+n}$ (since the left side can be obtained from the right one just by isotopy).

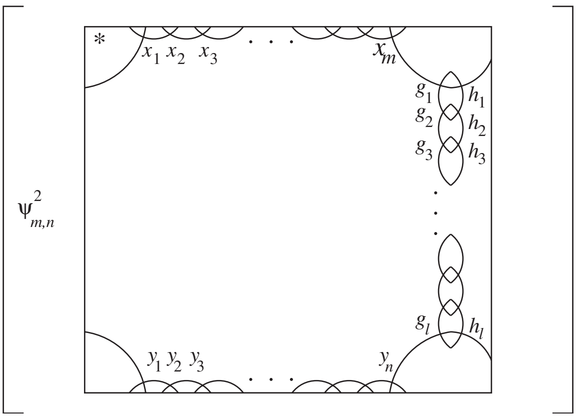
Lemma 2.1. *If* $\underline{x} \in G^m$, $\underline{y} \in G^n$ *for* $m, n \in \mathbb{N}$, *then*
(a) *for each* $\underline{g}$, $\underline{h} \in G^l$ *where* $l \in \mathbb{N}$, *we have*

$$[\psi^{2l}_{m,n}(T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}})] = \begin{cases} \delta^{(l-1)}[\psi^2_{m,n}(T^{\underline{x},\underline{y}}_{(g_1h_1^{-1}),1})] & if \ \underline{g} = (g_1h_1^{-1}) \cdot \underline{h}, \\ 0 & otherwise, \end{cases}$$

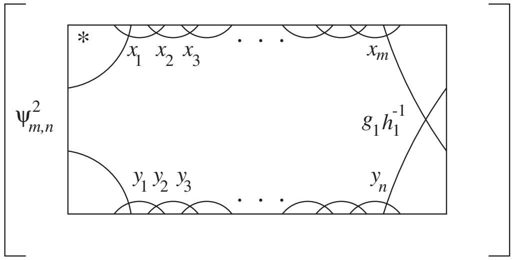
where we write $\delta = |G|^{\frac{1}{2}}$;
(b) *for* $g \in G$,

$$[\psi^2_{m,n}(T^{\underline{x},\underline{y}}_{g,1})] = [\psi^2_{m,n}(T^{\underline{x},g\cdot\underline{y}}_{g,1})] = [\psi^2_{m,n}(T^{\underline{x},s^g_{\underline{y}}}_{g,1})].$$

Proof. (a) First note that $[\psi^{2l}_{m,n}(T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}})] =$



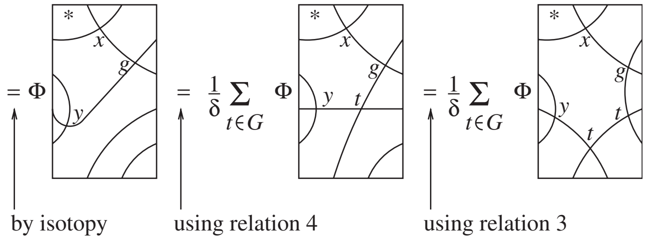
because one can pass the boxes labelled $h_1, h_2, \ldots, h_l$ in $T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}}$ from the left side of the internal box of $\Psi^{2l}_{m,n}$ to its right side, along the $2l$ strings around the internal disc (note that this equality holds even at the level of $(FAL)^m_n$). Further, using the relations 2 and 3 of the group planar algebra, the above element becomes zero if it is not the case that $g_1h_1^{-1} = g_2h_2^{-1} = \cdots = g_lh_l^{-1}$, equivalently $\underline{g} = (g_1h_1^{-1}) \cdot \underline{h}$; and in that case, it reduces to the following element:

$\delta^{(l-1)}$ 

thus proving (a).

(b) Using relations 3 and 4 of the group planar algebra, we can show that $\Phi((twist)_n^m(T^{\underline{x},\underline{y}}_{g,1})) = \frac{1}{\delta^n}\sum_{\underline{t}\in G^n}\Phi(T^{\underline{x},\underline{t}}_{(g,\underline{t}),(1,\underline{y})})$ for all $g \in G$. We exhibit the proof of this fact for the case $m = n = 1$.

$$\Phi((twist)_1^1(T^{x,y}_{g,1}))$$



$$= \delta^{-1}\sum_{t\in G}\Phi(T^{x,t}_{(g,t),(1,y)}).$$

So we have

$$\begin{aligned}
[\psi^2_{m,n}(T^{\underline{x},\underline{y}}_{g,1})] &= [\psi^{2n+2}_{m,n}((twist)^2_{m,n}(T^{\underline{x},\underline{y}}_{g,1}))] \quad \text{(by isotopy)}\\
&= \frac{1}{\delta^n}\sum_{\underline{t}\in G^n}[\psi^{2n+2}_{m,n}(T^{\underline{x},\underline{t}}_{(g,\underline{t}),(1,\underline{y})})]\\
&= \frac{1}{\delta^n}\sum_{\underline{t}\in G^n}\delta_{\underline{t},g\cdot\underline{y}}\,\delta^n[\psi^2_{m,n}(T^{\underline{x},\underline{t}}_{g,1})]\\
&= [\psi^2_{m,n}(T^{\underline{x},g\cdot\underline{y}}_{g,1})]
\end{aligned}$$

for all $\underline{x} \in G^m$, $\underline{y} \in G^n$, $g \in G$. Therefore,

$$[\psi^2_{m,n}(T^{\underline{x},\underline{y}}_{g,1})] = [\psi^2_{m,n}(T^{\underline{x},g^k\cdot\underline{y}}_{g,1})] = [\psi^2_{m,n}(T^{\underline{x},s^g_{\underline{y}}}_{g,1})]$$

for all $\underline{x} \in G^m$, $\underline{y} \in G^n$, $g \in G$, $k = 1, \cdots, o(g)$. $\square$

Remark 2.2. Proceeding along the line of arguments used in the proof of Lemma 2.1 (a), one can also prove:

for $\underline{x} \in G^m$, $x, y \in G$, $\underline{g}$, $\underline{h} \in G^l$,

$$[\psi^{2l}_{m,0^+}(T^{\underline{x},0^+}_{\underline{g},\underline{h}})] = \begin{cases} \delta^{(l-1)}[\psi^2_{m,0^+}(T^{\underline{x},0^+}_{(g_1h_1^{-1}),1})] & \text{if } \underline{g} = (g_1h_1^{-1})\cdot\underline{h},\\ 0 & \text{otherwise,}\end{cases}$$

$$[\psi^{2l}_{0^+,0^+}(T^{0^+,0^+}_{\underline{g},\underline{h}})] = \begin{cases} \delta^{(l-1)}[\psi^2_{0^+,0^+}(T^{0^+,0^+}_{(g_1h_1^{-1}),1})] & \text{if } \underline{g} = (g_1h_1^{-1})\cdot\underline{h}, \\ 0 & \text{otherwise,} \end{cases}$$

$$[\psi^{2l+1}_{m,0^-}(T^{\underline{x},y\{0^-\}}_{\underline{g},\underline{h}})] = \begin{cases} \delta^{(l-1)}[\psi^3_{m,0^-}(T^{\underline{x},y\{0^-\}}_{(g_1h_1^{-1}),1})] & \text{if } \underline{g} = (g_1h_1^{-1})\cdot\underline{h}, \\ 0 & \text{otherwise,} \end{cases}$$

$$[\psi^{2l+2}_{0^-,0^-}(T^{x\{0^-\},y\{0^-\}}_{\underline{g},\underline{h}})] = \begin{cases} \delta^{(l-1)}[\psi^4_{0^-,0^-}(T^{x\{0^-\},y\{0^-\}}_{(g_1h_1^{-1}),1})] & \text{if } \underline{g} = (g_1h_1^{-1})\cdot\underline{h} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.3. *For $m, n \in \mathbb{N}$, we have the following*:

(i) $(AP)^m_{0^+}$ *is linearly spanned by the subset*

$$\{[\psi^2_{m,0^+}(T^{\underline{x},0^+}_{g,1})] : \underline{x} \in G^m \text{ such that } x_1 = 1,\ g \in G\},$$

(ii) $(AP)^{0^+}_{0^+}$ *is linearly spanned by the subset*

$$\{[\psi^2_{0^+,0^+}(T^{0^+,0^+}_{g,1})] : g \in G\},$$

(iii) $(AP)^m_{0^-}$ *is linearly spanned by the subset*

$$\{[\psi^3_{m,0^-}(T^{\underline{x},1}_{g,1})] : \underline{x} \in G^m, g \in G\},$$

(iv) $(AP)^{0^-}_{0^-}$ *is linearly spanned by the subset*

$$\{[\psi^4_{0^-,0^-}(T^{x\{0^-\},1}_{g,1})] : x, g \in G\},$$

(v) $(AP)^m_n$ *is linearly spanned by the subset*

$$\{[\psi^2_{m,n}(T^{\underline{x},\underline{s}}_{g,1})] : \underline{x} \in G^m \text{ such that } x_1 = 1, g \in G, \underline{s} \in S^g_n\}.$$

Proof. (v) From the definition, it easily follows that any element of $(AP)^m_n$ is a linear combination of $[A]$'s for A in $(AnnL)^m_n$. Now, any $A \in (AnnL)^m_n$ being expressible as $\psi^l_{m,n}(T)$ for some $l \in 2\mathbb{N}$, $T \in \mathcal{T}_{l+m+n}(L)$, $(AP)^m_n$ is generated by $[\psi^l_{m,n}(T)]$'s for $l \in 2\mathbb{N}$, $T \in \mathcal{T}_{l+m+n}(L)$. Since $\Phi(T)$, for $T \in \mathcal{T}_{l+m+n}(L)$, is a linear combination of $\Phi(T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}})$ for $\underline{x} \in G^m$ such that $x_1 = 1$, $\underline{y} \in G^n$, $\underline{g}, \underline{h} \in G^{\frac{l}{2}}$, therefore the generating subset reduces to $\{[\psi^l_{m,n}(T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}})] : l \in 2\mathbb{N}, \underline{x} \in G^m$ such that $x_1 = 1, \underline{y} \in G^n, \underline{g}, \underline{h} \in G^{\frac{l}{2}}\}$. Now use parts (a) and (b) of Lemma 2.1, to get the desired result.

Proofs of (i)–(iv) are similar to that of (v), except that we do not need to use Lemma 2.1(b) and instead of Lemma 2.1(a), we use Remark 2.2. $\square$

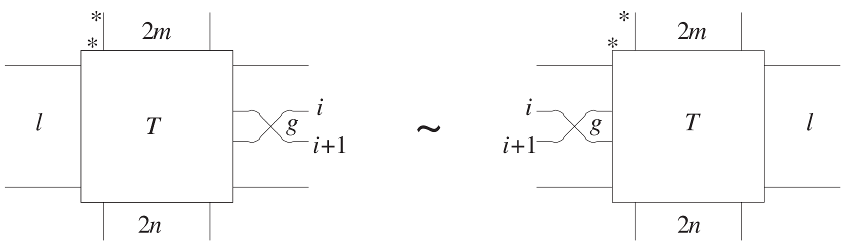
We first analyze $(AP)^m_n$ for $m, n \in \mathbb{N}$. It follows from Proposition 2.3(v) that the dimension of $(AP)^m_n$ is at most $|G|^{(m+n-1)} \sum_{g \in G} \frac{1}{o(g)}$. We will next prove that the dimension is actually equal to this expression, in other words, that the subset mentioned in Proposition 2.3(v) forms a basis of $(AP)^m_n$. For this, let V^m_n be the subspace of P_{m+n+2}, with $\{\Phi(T^{\underline{x},\underline{s}}_{g,1}) : \underline{x} \in G^m$ such that $x_1 = 1, g \in G, \underline{s} \in S^g_n\}$ as basis. Now we define a map $\sim : \amalg_{l \in 2\mathbb{N}} \mathcal{P}_{m+n+l}(L) \to V^m_n$ (where $\amalg$ denotes disjoint union) by the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{P}_{m+n+l}(L) & \xrightarrow{\Phi} & P_{m+n+l} \\ \sim\downarrow & \swarrow \mu & \\ V^m_n & & \end{array}$$

for each $l \in 2\mathbb{N}$, where $\mu : P_{m+n+l} \to V^m_n$ is the linear map sending $\Phi(T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}})$ to $\left[\delta_{\underline{g},(g_1h_1^{-1})\cdot\underline{h}}\ \delta^{(\frac{l}{2}-1)}\ \Phi(T^{\underline{x},\underline{s}_{\underline{y}}^{(g_1h_1^{-1})}}_{(g_1h_1^{-1}),1})\right]$ for $\underline{x} \in G^m$ such that $x_1 = 1$, $\underline{y} \in G^n$, $\underline{g}, \underline{h} \in G^{\frac{l}{2}}$.

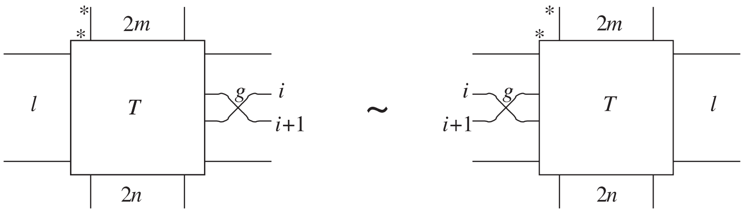
We now introduce some more notation. Let $\mathcal{S} = \amalg_{l \in 2\mathbb{N}} \mathcal{T}_{m+n+l}(L)$. Consider the equivalence relation $\sim$ on $\mathcal{S}$ generated by the following relations, for $T \in \mathcal{T}_{m+n+l}(L) \subset \mathcal{S}$:

(i)$'$

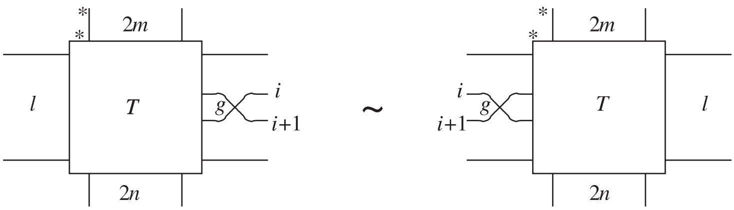


where the i and $(i+1)$ on the right (resp. left) side of the tangle on the left (resp. right) denotes the ith and $(i+1)$th string counting from the topmost string coming out from the right (resp. left) side of T. (This convention will be followed in the following argument.)

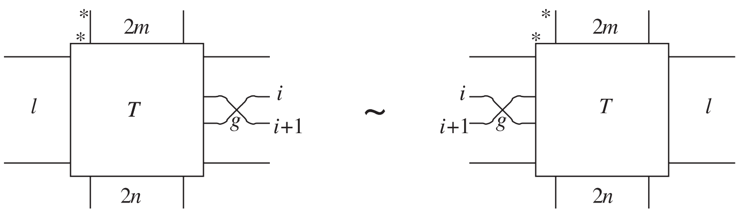
(i)$''$



(i)$'''$

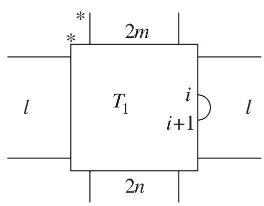


(i)$''''$

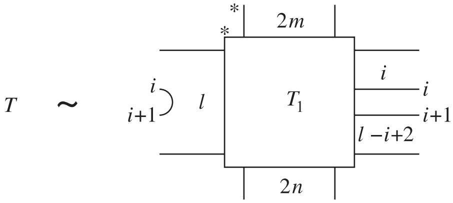


(ii)$'$ suppose $T_1 \in \mathcal{T}_{m+n+l+1}(L)$ is such that

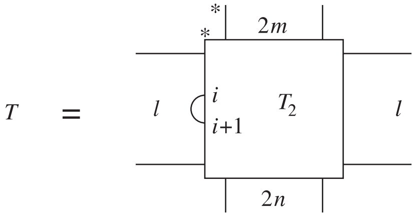
$T \quad =$



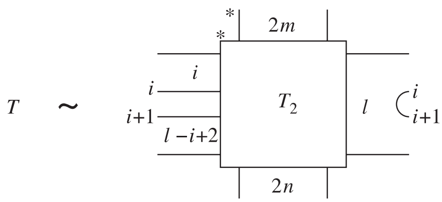
then



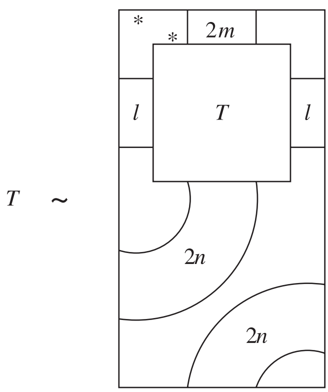
(ii)″ if $T_2 \in \mathcal{T}_{m+n+l+1}(L)$ is such that



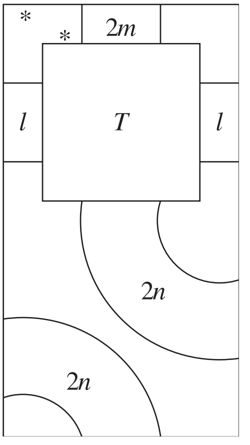
then



(iii)′



(iii)″

$T \quad \sim$ 

Remark 2.4. Using isotopy, one can easily show that if $\mathcal{T}_{m+n+l}(L) \ni T \sim T' \in \mathcal{T}_{m+n+l'}(L)$, then $\psi^{l}_{m,n}(T) = \psi^{l'}_{m,n}(T')$. The reverse implication is also true and the proof is similar to the arguments used in Proposition 2.8.

Remark 2.5. Relations (iii)′ and (iii)″ correspond to the twisting of the internal disc (of the annular tangle $\psi^{l}_{m,n}(T)$) by 360°. Note that we do not demand the same thing for the external boundary since they can be obtained using the relations. In later sections, we will see how the relations of type (iii) are responsible for making the algebra $(AP)^1_1$ isomorphic to a quotient of the quantum double of a group by a nontrivial ideal.

Remark 2.6. Note that we do not use anything special of the group planar algebras for defining $\mathcal{S}$ and $\sim$. The set $\mathcal{S}$ and the relation can be generalized for any finitely generated planar algebras by replacing 'g' in the relations of type (i) with labelled discs.

Lemma 2.7. *If $T \sim T'$ for $T \in \mathcal{T}_{m+n+l}(L)$, $T' \in \mathcal{T}_{m+n+l'}(L)$, $l, l' \in 2\mathbb{N}$, then $\widetilde{T} = \widetilde{T}'$.*

Proof. The proof is a straight forward application of the relations of the group planar algebra and Lemma 2.1. For example, let us prove the lemma in the case T and T' are related by (iii)′. Without loss of generality, let T be $T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}}$ where $\underline{x} \in G^m$ such that $x_1 = 1$, $\underline{y} \in G^n$ and $\underline{g}, \underline{h} \in G^{\frac{l}{2}}$ and T' be the tangle on the right side of $\sim$ in relation (iii)′. (This is because (a) elements of the form $\Phi(T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}})$'s generate P_{m+n+l}, (b) the map $\sim: \mathcal{P}_{m+n+l}(L) \to V^m_n$ factors through P_{m+n+l} via the presenting map Φ and (c) T' arises from an annular action on T.) Using arguments similar to that in the first part of the proof of Lemma 2.1(b), we have

$$\Phi(T') = \frac{1}{\delta^n} \sum_{\underline{t} \in G^n} \Phi(T^{\underline{x},\underline{t}}_{(\underline{g},\underline{t}),(\underline{h},\underline{y})}).$$

Now

$$\tilde{T}' = \mu(\Phi(T')) = \frac{1}{\delta^n} \sum_{\underline{t}\in G^n} \mu(\Phi(T^{\underline{x},\underline{t}}_{(\underline{g},\underline{t}),(\underline{h},\underline{y})}))$$

$$= \frac{1}{\delta^n}\ \delta_{\underline{g},(g_1h_1^{-1})\cdot\underline{h}}\ \delta^{(n+\frac{l}{2}-1)}\ \Phi(T^{\underline{x},\underline{s}^{(g_1h_1^{-1})}_{(g_1h_1^{-1})\cdot\underline{y}}}_{(g_1h_1^{-1}),1}) \quad \text{(by definition of } \mu)$$

$$= \delta_{\underline{g},(g_1h_1^{-1})\cdot\underline{h}}\ \delta^{(\frac{l}{2}-1)}\ \Phi(T^{\underline{x},\underline{s}^{(g_1h_1^{-1})}_{\underline{y}}}_{(g_1h_1^{-1}),1}) \quad \text{(by definition of } S_n^{(g_1h_1^{-1})})$$

$$= \tilde{T}. \qquad \square$$

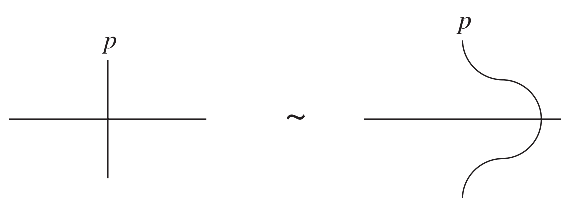
Proposition 2.8. *If* $\psi^l_{m,n}(T) = \psi^{l'}_{m,n}(T')$ *for* $T \in \mathcal{T}_{m+n+l}(L)$, $T' \in \mathcal{T}_{m+n+l'}(L)$, $l, l' \in 2\mathbb{N}$, *then* $\widetilde{T} = \widetilde{T}'$.

Proof. In view of Lemma 2.7, if we prove that $\psi^l_{m,n}(T) = \psi^{l'}_{m,n}(T') = A$ (say) implies $T \sim T'$, then we are through.

Fix an element in the isotopy class of A and call it A_1. If $A = \psi^l_{m,n}(T)$, then T can be obtained by cutting A_1 along a path $p \in \mathcal{P}(A_1)$, where we shall write $\mathcal{P}(A_1)$ for the collection of directed simple paths p in the annulus A_1 which satisfy the following conditions: (i) p starts at $*_{\text{int}}$ and ends at $*_{\text{ext}}$, where $*_{\text{int}}$ (resp. $*_{\text{ext}}$) denotes a point on the segment of the internal (resp. external) boundary of A_1, lying strictly between the first and the last point, and (ii) p meets the strings and the boundaries of A_1 transversely, and intersects at least one string of A_1. After cutting along p, we can view the annulus A_1 as the tangle T, provided we take the first point of the external boundary of A_1 as the first point of T.

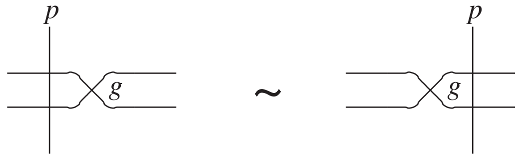
Motivated by the equivalence on $\mathcal{S}$, we define the equivalence relation on $\mathcal{P}(A_1)$ generated by the following 'local moves':

(o)′

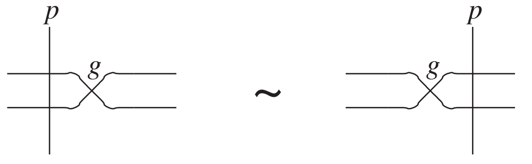


(o)″

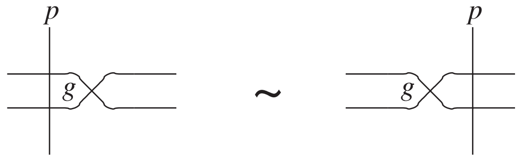
(i)′



(i)″

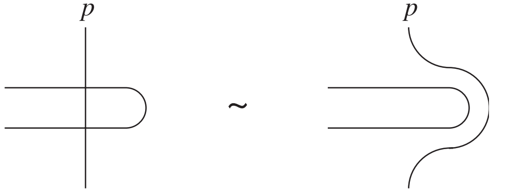


(i)‴

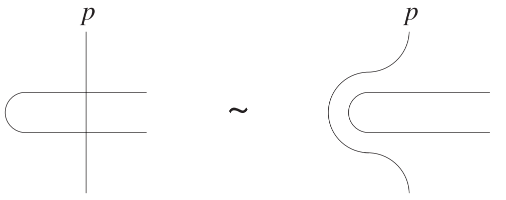


(i)⁗

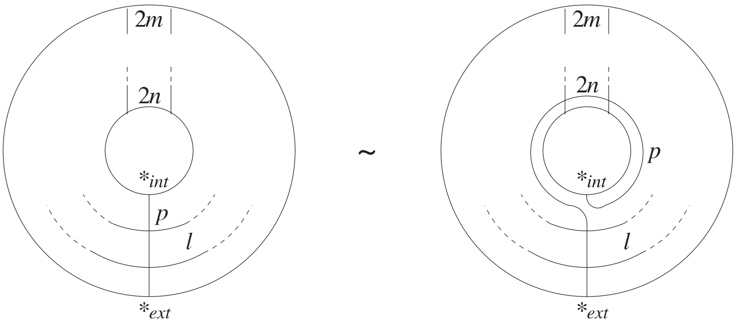
(ii)′



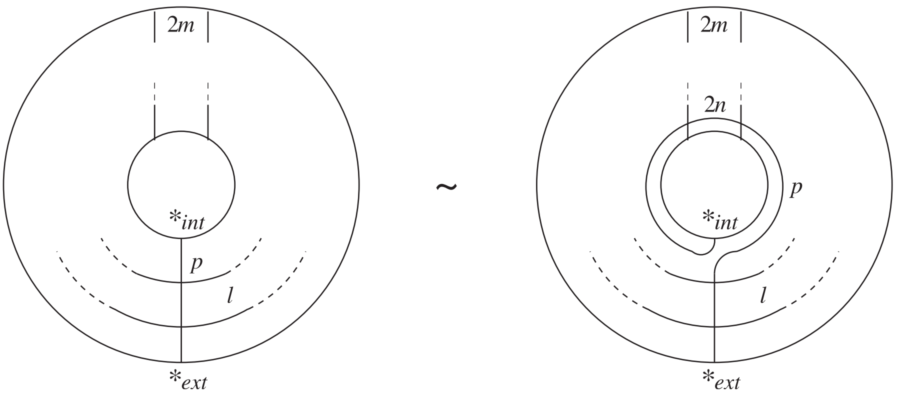
(ii)″



(iii)′

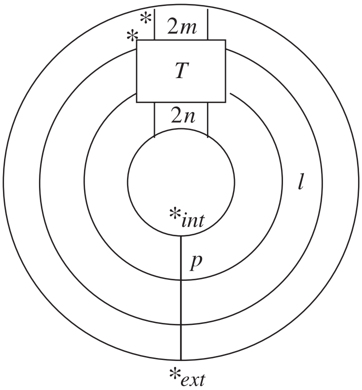


(iii)″

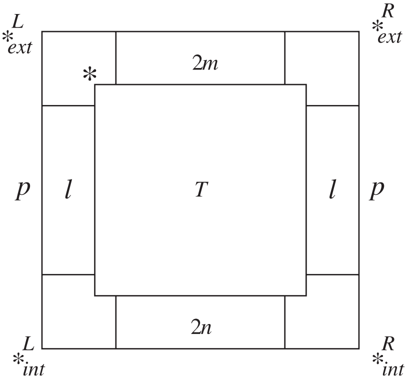


We remark that two paths related by (o) moves give the same tangle in $\mathcal{S}$, that is, the (o) moves correspond to isotopy of tangles; also the moves (i), (ii) and (iii) correspond to the relations (i), (ii) and (iii) in $\mathcal{S}$, respectively. Therefore, if $p, p' \in \mathcal{P}(A_1)$ satisfy $p \sim p'$ and if T and T' are the tangles in $\mathcal{S}$ given by cutting along p and p', respectively, then $T \sim T'$.

Suppose p (resp. p') is a path in $\mathcal{P}(A_1)$ from which T (resp. T') is obtained. From the above remark, it is enough, in order to complete the proof of the proposition, to prove that $p \sim p'$. Without loss of generality, we may assume that A_1 looks like:

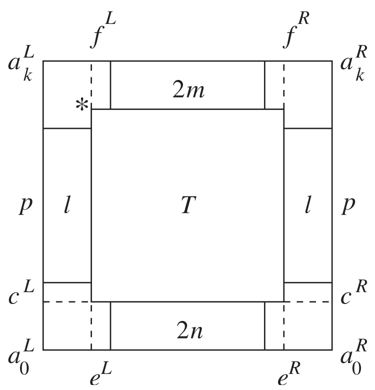


After cutting along p, A_1 looks like



The superscripts L, R of $*_{\text{int}}$ and $*_{\text{ext}}$ denote left and right, respectively. In A_1, $*^{\text{L}}_{\text{ext}}$ and $*^{\text{R}}_{\text{ext}}$ (resp. $*^{\text{L}}_{\text{int}}$ and $*^{\text{R}}_{\text{int}}$) are identified and also the sides $(*^{\text{L}}_{\text{int}}*^{\text{L}}_{\text{ext}})$ and $(*^{\text{R}}_{\text{int}}*^{\text{R}}_{\text{ext}})$ along the path p. This identification is done in such a way that the side $(*^{\text{L}}_{\text{ext}}*^{\text{R}}_{\text{ext}})$ forms the external boundary of A_1. We also assume that the sides of the rectangular boundary of T are either parallel or perpendicular to the sides of the rectangle $(*^{\text{L}}_{\text{int}}*^{\text{R}}_{\text{int}}*^{\text{R}}_{\text{ext}}*^{\text{L}}_{\text{ext}})$ and that the strings outside T are straight lines and meet the boundaries of the rectangles perpendicularly. Further we assume that the distance between the side $(*^{\text{L}}_{\text{int}}*^{\text{R}}_{\text{int}})$ and the ith string coming from the left of T, counted from below is equal to that from the right for $i = 1, \cdots, l$. In the forthcoming argument, we will go back and forth between the annular picture and the rectangular view of A_1. Any point ω on p in the annular view will correspond to points ω^{L} and ω^{R} on $(*^{\text{L}}_{\text{int}}*^{\text{L}}_{\text{ext}})$ and $(*^{\text{R}}_{\text{int}}*^{\text{R}}_{\text{ext}})$, respectively.

Next consider the path p'. Without loss of generality, we may assume that p' intersects p transversely. Let $*_{\mathrm{int}} = a_0, a_1, \ldots, a_k = *_{\mathrm{ext}}$ be the points of intersection of p and p' with the order induced by p. In the rectangular view of A_1, p' becomes a disjoint union of segments from either a_i^{L} or a_i^{R} to a_{i+1}^{L} or a_{i+1}^{R} for $i = 0, 1, 2, \ldots, (k-1)$. These segments of p' are of two kinds, namely, starting and ending on (i) the same side, or (ii) different sides. We extend the bottom and two vertical sides of the rectangular boundary of T in both directions to meet the boundary of the rectangle ($*_{\mathrm{int}}^{\mathrm{L}}\, *_{\mathrm{int}}^{\mathrm{R}}\, *_{\mathrm{ext}}^{\mathrm{R}} *_{\mathrm{ext}}^{\mathrm{L}}$) at $c^{\mathrm{L}}, c^{\mathrm{R}}, e^{\mathrm{L}}, f^{\mathrm{L}}, e^{\mathrm{R}}, f^{\mathrm{R}}$, say, so that the rectangular version of A_1 now looks like this:



□

Assertion. *There exists a path $q \in \mathcal{P}(A_1)$, intersecting p transversely such that*

(i) $q \sim p'$,
(ii) *q intersects p exactly at $(k+1)$ many points, say,* $*_{\mathrm{int}} = b_0, b_1, \ldots, b_k = *_{\mathrm{ext}}$,
(iii) $\{b_i^{\mathrm{L}}\}_{i=1}^{k-1}$ *(resp.* $\{b_i^{\mathrm{R}}\}_{i=1}^{k-1}$*) lies on the interval* $[a_0^{\mathrm{L}}, c^{\mathrm{L}}]$ *(resp.* $[a_0^{\mathrm{R}}, c^{\mathrm{R}}]$*) and the segment of q joining* $b_{k-1}^{\mathrm{L/R}}$ to $a_k^{\mathrm{L/R}}$ *lies inside the union of the rectangles* $(a_0^{\mathrm{L}} a_k^{\mathrm{L}} f^{\mathrm{L}} e^{\mathrm{L}})$, $(a_0^{\mathrm{L}} c^{\mathrm{L}} c^{\mathrm{R}} a_0^{\mathrm{R}})$ *and* $(e^{\mathrm{R}} f^{\mathrm{R}} a_k^{\mathrm{R}} a_0^{\mathrm{R}})$.

It is not hard to see that if such q exists, then $q \sim p$ using moves (o), (ii) *and* (iii). *So it suffices to prove the assertion.*

Proof. We shall first inductively construct paths q_j, for $0 \leqslant j < k$ satisfying the following conditions:

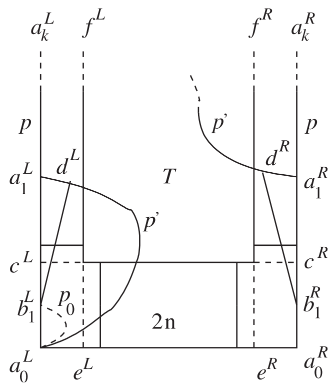
(i) $q_j \in \mathcal{P}(A_1)$,
(ii) $q_j \sim p'$,
(iii) q_j intersects p at exactly $(k+1)$ points, say, at $b_i(q_j)$, $0 \leqslant i \leqslant k$, with $b_0(q_j) = *_{\mathrm{int}}$ and $b_k(q_j) = *_{\mathrm{ext}}$, and such that for $0 \leqslant i \leqslant j$, the point $b_i(q_j)$ lies on the segment of p between a_0 and c,
(iv) the segment of q_j between $b_{j+1}(q_j)$ and $b_k(q_j)$ $(= *_{\mathrm{ext}})$ coincides with the segment of p' between a_{j+1} and a_k, and

(v) the segment of q_j between $b_0(q_j)$ and $b_j(q_j)$ lies below the straight line joining $b_j^{\mathrm{L}}(q_j)$ and $b_j^{\mathrm{R}}(q_j)$.

Finally, we shall 'doctor' the path q_{k-1} to obtain the path q of the assertion.

We set the induction rolling by setting $q_0 = p'$. Next, consider the case $j = 1$. If the segment of p' joining a_0 and a_1 lies strictly below the straight line $(c^{\mathrm{L}}c^{\mathrm{R}})$, put $q_1 = p'$. So, assume the segment of p' joining a_0 and a_1 does not lie strictly below the straight line $(c^{\mathrm{L}}c^{\mathrm{R}})$. Without loss of generality, we may assume that p' starts from a_0^{L}. We consider the following two cases.

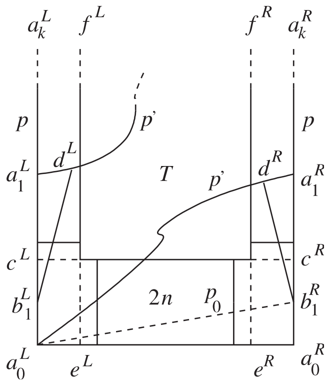
Case 1: The segment of p', starting from a_0^{L}, ends at a_1^{L}, that is, on the same side.



Choose a point b_1^{L} (resp. b_1^{R}) on the open interval $(a_0^{\mathrm{L}}c^{\mathrm{L}})$ (resp. $(a_0^{\mathrm{R}}c^{\mathrm{R}})$) such that $|(a_0^{\mathrm{L}}\ b_1^{\mathrm{L}})| = |(a_0^{\mathrm{R}}\ b_1^{\mathrm{R}})|$ as well as a simple curve p_0 from a_0^{L} to b_1^{L} such that p_0 lies below the straight line joining b_1^{L} and b_1^{R}, never touches the straight line $(e^{\mathrm{L}} f^{\mathrm{L}})$ and meets p as well as $(a_0^{\mathrm{L}}a_0^{\mathrm{R}})$ transversely. Choose a point d^{L} (resp. d^{R}) on p', very close to a_1^{L} (resp. a_1^{R}), lying in the rectangle $(a_0^{\mathrm{L}}a_k^{\mathrm{L}} f^{\mathrm{L}}e^{\mathrm{L}})$ (resp. $(e^{\mathrm{R}} f^{\mathrm{R}}a_k^{\mathrm{R}}a_0^{\mathrm{R}})$) and join it to b_1^{L} (resp. b_1^{R}) by a straight line (as in the above picture). Consider the path $\widetilde{q_1}$ in A_1 given by p' from a_0^{L} to d^{L}, then along the straight line from d^{L} to b_1^{L}, followed by the straight line from b_1^{R} to d^{R} and the rest along p' from d^{R} to a_k $(= *_{\mathrm{ext}})$. Clearly, $\widetilde{q_1} \in \mathcal{P}(A_1)$. Moreover, the segment $(d^{\mathrm{L}}b_1d^{\mathrm{R}})$ along $\widetilde{q_1}$ in A_1 can be obtained from the segment $(d^{\mathrm{L}}a_1d^{\mathrm{R}})$ along p' by moves (o) and (ii), where b_1 is the point on p in A_1 corresponding to $b_1^{\mathrm{L/R}}$ in the rectangular version. Since $\widetilde{q_1}$ and p' coincide except between d^{L} and d^{R}, we may conclude that indeed $\widetilde{q_1} \sim p'$.

Now consider the path q_1 in A_1 given by p_0 from a_0^{L} to b_1^{L} and the rest along $\widetilde{q_1}$ from b_1^{R} to a_k. Note that $\widetilde{q_1} \in \mathcal{P}(A_1)$. The segment between a_0^{L} and b_1^{L} along q_1 in A_1 can be obtained from that along $\widetilde{q_1}$ by moves (o), (i) and (ii). Since q_1 and $\widetilde{q_1}$ match everywhere except between a_0 and b_1, we have $q_1 \sim \widetilde{q_1} \sim p'$. Also q_1 intersects p at exactly $(k+1)$ points and the segment between a_0 $(= b_0(q_1))$ and b_1 $(= b_1(q_1))$ along q_1 lies below the straight line $(b_1^{\mathrm{L}}(q_1)b_1^{\mathrm{R}}(q_1))$.

Case 2: The segment of p', starting from a_0^{L} ends at a_1^{R}, that is, on the different side.



Choose a point b_1^{L} (resp. b_1^{R}) on the open interval $(a_0^{\mathrm{L}}c^{\mathrm{L}})$ (resp. $(a_0^{\mathrm{R}}c^{\mathrm{R}})$) such that $|(a_0^{\mathrm{L}}\ b_1^{\mathrm{L}})| = |(a_0^{\mathrm{R}}\ b_1^{\mathrm{R}})|$ as well as a simple curve p_0 from a_0^{L} to b_1^{R} such that p_0 touches p, $(a_0^{\mathrm{L}}a_0^{\mathrm{R}})$ and strings coming out of T transversely and lies below the straight line joining b_1^{L} and b_1^{R}. Choose a point d^{L} (resp. d^{R}) on p', very close to a_1^{L} (resp. a_1^{R}), lying in the rectangle $(a_0^{\mathrm{L}}a_k^{\mathrm{L}}f^{\mathrm{L}}e^{\mathrm{L}})$ (resp. $(e^{\mathrm{R}}f^{\mathrm{R}}a_k^{\mathrm{R}}a_0^{\mathrm{R}})$) and join it to b_1^{L} (resp. b_1^{R}) by a straight line (as in the above picture). Consider the path $\widetilde{q_1}$ in A_1 given by p' from a_0^{L} to d^{R}, then along the straight line from d^{R} to b_1^{R}, followed by the straight line from b_1^{L} to d^{L} and the rest along p' from d^{L} to a_k $(= *_{\mathrm{ext}})$. Clearly, $\widetilde{q_1} \in \mathcal{P}(A_1)$. Moreover, the segment $(d^{\mathrm{R}}b_1d^{\mathrm{L}})$ along $\widetilde{q_1}$ in A_1 can be obtained from the segment $(d^{\mathrm{R}}a_1d^{\mathrm{L}})$ along p' by moves (o) and (ii), where b_1 is the point on p in A_1 corresponding to $b_1^{\mathrm{L/R}}$ in the rectangular version. Since $\widetilde{q_1}$ and p' coincide except between d^{L} and d^{R}, we may conclude that indeed $\widetilde{q_1} \sim p'$.

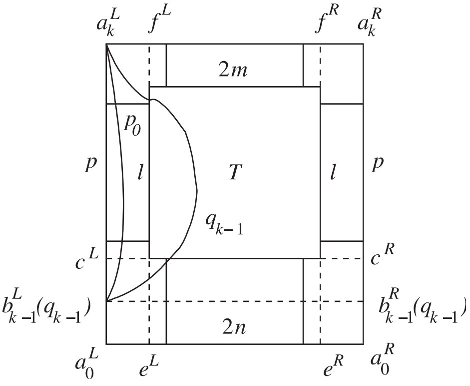
Now consider the path q_1 in A_1 given by p_0 from a_0^{L} to b_1^{R} and the rest along $\widetilde{q_1}$ from b_1^{R} to a_k. Note that $q_1 \in \mathcal{P}(A_1)$. The segment between a_0^{L} and b_1^{R} along q_1 in A_1 can be obtained from that along $\widetilde{q_1}$ by moves (o), (i) and (ii). Since q_1 and $\widetilde{q_1}$ match everywhere except between a_0 and b_1, we have $q_1 \sim \widetilde{q_1} \sim p'$. Also q_1 intersects p at exactly $(k+1)$ points and the segment between a_0 $(= b_0(q_1))$ and b_1 $(= b_1(q_1))$ along q_1 lies below the straight line $(b_L(q_1)b_R(q_1))$.

Thus, in either case, q_1 satisfies conditions (i)–(v) (for $j = 1$).

Now suppose $2 \leqslant (j+1) \leqslant (k-1)$ and we have obtained a path q_j satisfying conditions (i)–(v). Following the construction of q_1 from p', one can obtain q_{j+1} from q_j, with $b_j(q_j)$ playing the role of $*_{\mathrm{int}}$, such that q_{j+1} satisfies all the above conditions with j replaced by $(j+1)$.

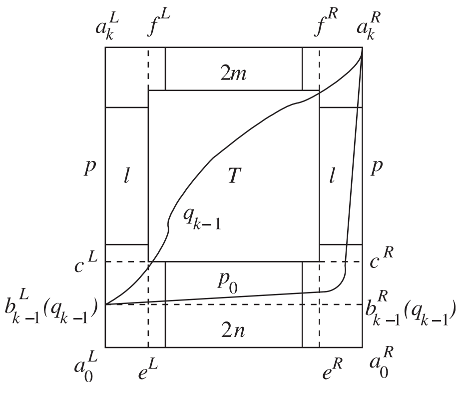
Without loss of generality, we may assume that the segment of q_{k-1}, between $b_{k-1}(q_{k-1})$ and $*_{\mathrm{ext}}$, starts from $b_{k-1}^{\mathrm{L}}(q_{k-1})$. We consider the following two cases:

Case (a): q_{k-1} ends at a_k^{L} $(= *_{\mathrm{ext}}^{\mathrm{L}})$.



Choose a simple curve p_0 from $b^{\mathrm{L}}_{k-1}(q_{k-1})$ to a^{L}_k, lying in the rectangle $(e^{\mathrm{L}}a^{\mathrm{L}}_0 a^{\mathrm{L}}_k f^{\mathrm{L}})$, intersecting the strings coming out of T transversely, and meeting p transversely and only at two points (as in the above picture). Let q be the path in A_1 given by q_{k-1} from a_0 to $b_{k-1}(q_{k-1})$ and then from $b_{k-1}(q_{k-1})$ to a_k along p_0. Note that p_0 can be obtained from the segment of q_{k-1} between $b_{k-1}(q_{k-1})$ and a_k using moves (o), (i) and (ii). Since q coincides with q_{k-1} everywhere except between $b_{k-1}(q_{k-1})$ and a_k, therefore $q \sim q_{k-1} \sim p'$. It is easy to see that q also satisfies conditions (ii) and (iii) of the assertion.

Case (b): q_{k-1} ends at a^{R}_k $(= *^{\mathrm{R}}_{\mathrm{ext}})$.



Choose a simple curve p_0 from $b^{\mathrm{L}}_{k-1}(q_{k-1})$ to a^{R}_k, lying in the union of the two rectangles, $(b^{\mathrm{L}}_{k-1}(q_{k-1})c^{\mathrm{L}}c^{\mathrm{R}}b^{\mathrm{R}}_{k-1}(q_{k-1}))$ and $(e^{\mathrm{R}}a^{\mathrm{R}}_0 a^{\mathrm{R}}_k f^{\mathrm{R}})$, intersecting the strings coming out of T transversely, and meeting p transversely and only at two points (as in the above picture). Let q be the path in A_1 given by q_{k-1} from a_0 to $b_{k-1}(q_{k-1})$ and then from $b_{k-1}(q_{k-1})$ to a_k along p_0. Note that p_0 can be obtained from the segment of q_{k-1} between $b_{k-1}(q_{k-1})$ and a_k using moves (o), (i) and (ii). Since q coincides with

q_{k-1} everywhere except between $b_{k-1}(q_{k-1})$ and a_k, therefore $q \sim q_{k-1} \sim p'$. It is easy to see that q also satisfies conditions (ii) and (iii) of the assertion.

Hence the assertion, and also the proposition, is established. □

Next, we define a map $\Gamma : (FAL)^m_n \to V^m_n$, first on the basis $(AnnL)^m_n$, and then extend linearly. Let $A \in (AnnL)^m_n$. We already know that there exist $T \in \mathcal{T}_{m+n+l}(L)$ (not necessarily unique) such that $A = \psi^l_{m,n}(T)$ for some $l \in 2\mathbb{N}$. Define $\Gamma(A)$ as $\widetilde{T}$. By Proposition 2.8, Γ is well defined.

Remark 2.9. The following diagram commutes for each $l \in 2\mathbb{N}$:

$$\begin{array}{ccc} \mathcal{P}_{m+n+l}(L) & \stackrel{\psi^l_{m,n}}{\longrightarrow} & (FAL)^m_n \\ \sim\downarrow & \swarrow\Gamma & \\ V^m_n & & \end{array}$$

(Reason: Since all the three maps are linear, it is enough to check the image of $\mathcal{T}_{m+n+l}(L)$.)

Proposition 2.10. *Γ induces an isomorphism from $(AP)^m_n$ to V^m_n, which maps $[X]$ to $\widetilde{X}$ for $X \in (FAL)^m_n$.*

Proof. $\Gamma(\psi^2_{m,n}(T^{\underline{x},\underline{s}}_{g,1})) = \widetilde{T^{\underline{x},\underline{s}}_{g,1}} = \Phi(T^{\underline{x},\underline{s}}_{g,1})$ for all $\underline{x} \in G^m$ such that $x_1 = 1$, $g \in G$, $\underline{s} \in S^g_n$, therefore Γ is surjective.

Now, suppose $B \in \mathcal{W}(L)^m_n$. Then there exists $l \in 2\mathbb{N}$ and $X \in \mathcal{P}_{m+n+l}(L) \cap kernel(\Phi)$ such that $B = \psi^l_{m,n}(X)$. From Remark 2.9, we have $\Gamma(B) = \Gamma(\psi^l_{m,n}(X)) = \widetilde{X} = \mu(\Phi(X)) = 0$. Thus $\mathcal{W}(L)^m_n \subset kernel(\Gamma)$.

Thus Γ induces a surjective linear map from the quotient $(AP)^m_n$ to V^m_n. From Proposition 2.3(v), we find the dimension of $(AP)^m_n$ is at most that of V^m_n. Hence the surjection induced by Γ is an isomorphism. □

Thus we have proved part (v) of the following (main) theorem of this section.

Theorem 2.11. *For $m, n \in \mathbb{N}$, we have the following*:

(i) $\{[\psi^2_{m,0^+}(T^{\underline{x},0^+}_{g,1})] : g \in G, \underline{x} \in G^m$ *such that* $x_1 = 1\}$ *forms a basis of* $(AP)^m_{0^+}$,

(ii) $\{[\psi^2_{0^+,0^+}(T^{0^+,0^+}_{c,1})] : c \in \mathcal{C}\}$ *forms a basis of* $(AP)^{0^+}_{0^+}$ *where $\mathcal{C}$ is a set of representatives of the conjugacy classes of G*,

(iii) $\{[\psi^1_{m,0^-}(T^{\underline{x},0^-})] : \underline{x} \in G^m\}$ *forms a basis of* $(AP)^m_{0^-}$,

(iv) $\{[\psi^2_{0^-,0^-}(T^{0^-,g,0^-})] : g^{0^-} \in G)\}$ *forms a basis of* $(AP)^{0^-}_{0^-}$,

(v) $\{[\psi^2_{m,n}(T^{\underline{x},\underline{s}}_{g,1})] : \underline{x} \in G^m$ *such that* $x_1 = 1, g \in G, \underline{s} \in S^g_n\}$ *forms a basis of* $(AP)^m_n$.

Proof. We have already proved (v). For (i)–(iv), we will make use of the spanning set obtained in Proposition 2.3 to get the desired set in this theorem. Proof of linear independence is similar to that of (v) by considering equivalence on the set of tangles, we leave this to the reader; however, we remark that the set of tangles, $\mathcal{S}$ will no longer need the equivalence corresponding to the rotation of the internal boundary by 360° because of the absence strings attached to the internal boundary in the cases, (i)–(iv).

(i) This is already proved by Proposition 2.3(i).

(ii) Note that for $g, h \in G$, $[\psi^2_{0^+,0^+}(T^{0^+,0^+}_{hgh^{-1},1})] = [\psi^2_{0^+,0^+}(T^{0^+,0^+}_{g,1})]$ since we can make h pass from the left side to the right side along the strings around the internal disc. Thus, by Proposition 2.3(ii), the subset mentioned in (ii) above indeed spans $(AP)^{0^+}_{0^+}$.

(iii) A careful observation of the annular tangle $\psi^3_{m,0^-}(T^{\underline{x},1}_{g,1})$ for $\underline{x} \in G^m$, $g \in G$, will tell us that we have a conditional expectation on g and hence we can assume g to be 1 and then the corresponding element of $(AP)^m_{0^-}$ is just a non-zero scalar times $[\psi^1_{m,0^-}(T^{\underline{x},0^-})]$. Now, use Proposition 2.3 (iii), to show that the set in (iii) above generates $(AP)^m_{0^-}$.

(iv) Proof of this is exactly similar to (iii) where m is replaced by 0^-. $\square$

Remark 2.12. The basis of $(AP)^{0^\pm}_n$ can be obtained just by considering the involution, $*$.

Remark 2.13. Arguments used in proving (iii) will establish that $(AP)^{0^\mp}_{0^\pm}$ is spanned by a single element $[A^{0^\mp}_{0^\pm}]$ where $A^{0^\mp}_{0^\pm}$ is an annular $(0^\mp, 0^\pm)$-tangle with just a single loop enclosing the internal disc, respectively.

2.2. Irreducible representations of the group planar algebra

We first observe that every element of $(AP)^m_n$, for $m, n \geqslant 2$, can be written as a linear combination of composition an element in $(AP)^m_1$ with an element in $(AP)^1_n$. For instance,

$$[\psi^2_{m,n}(T^{\underline{x},\underline{y}}_{g,1})] = \sum_{t \in G} [\psi^2_{m,1}(T^{\underline{x},1}_{g,1})] \cdot [\psi^2_{1,n}(T^{1,\underline{y}}_{t,1})]$$

for all $\underline{x} \in G^m$, $\underline{y} \in G^n$, $g \in G$. This means that $\widehat{(AP)_k} = (AP)^k_k$, for all $k \geqslant 2$ and $(LWP)_k = \{0\}$. Thus, we have the following lemma.

Lemma 2.14. *There are no irreducible $*$-representation with weight greater than or equal to* 2.

So we can restrict ourselves to representations of weight $0^\pm$ or 1, where of course we say that a representation F has weight 0^+ (resp., 0^-) if $F(0^+) \neq 0$ (resp., $F(0^-) \neq 0$). Before going into more details, let us fix some notations. Let $\mathcal{C}$ be the set of conjugacy classes of G. For each $C \in \mathcal{C}$, fix an element $c \in C$ and let o_C be the order of c. Let N_C be the stabilizer of c in G, that is, $N_C = \{g \in G : gc = cg\}$, and $n_C := |N_C|$. Let D_C be the set of elements in G representing distinct left cosets of N_C such that $1 \in D_C$. Therefore, $|D_C| = |C|$ and

$$\amalg_{d \in D_C} N_C \ d = G.$$

Let E_C be a set of elements in N_C representing distinct left cosets of $\langle c \rangle$, the cyclic group generated by c. Therefore,

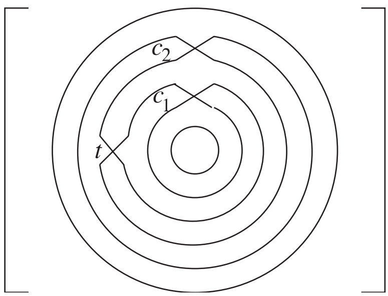
$$\amalg_{e \in E_C} \langle c \rangle e = N_C \quad \text{and} \quad \amalg_{e \in E_C, d \in D_C} \langle c \rangle ed = G$$

and thus $\{ed : e \in E_C, d \in D_C\}$ is a set of elements of G which represent distinct left cosets of $\langle c \rangle$ in G.

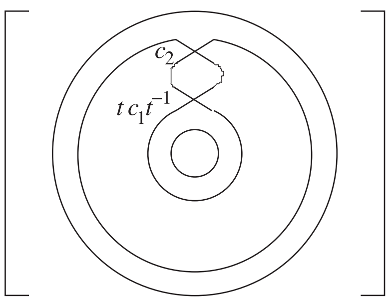
We now proceed towards describing the algebraic structure for $k = 0^{\pm}, 1$.

Case 0^+ : For $C \in \mathcal{C}$, define $E_C^{0^+}$ as $n_C^{-1}[\psi^2_{0^+,0^+}(T_{c,1}^{0^+,0^+})]$. If $C_1, C_2 \in \mathcal{C}$, then $E_{C_1}^{0^+} \cdot E_{C_2}^{0^+}$ is equal to:

$$\frac{\delta^{-1}}{n_{C_1} n_{C_2}} \sum_{t \text{ in } G}$$



$$= \frac{\delta^{-1}}{n_{C_1} n_{C_2}} \sum_{t \text{ in } G}$$



which is zero unless $C_1 = C_2$, and in the case $C_1 = C_2$, it is equal to $E_{C_1}^{0^+}$.

In order to arrive at the step indicated by the first diagram above, as well as numerous places in this paper, we use the relation in the group planar algebra which relates the Jones projection e_1 to the average of the (unitaries corresponding to the) group elements. Similarly, the reason that the expression given by the second diagram vanishes unless two specific group elements are conjugate is a consequence of another relation in the group planar algebra. (We will not give such elaborate explanations when we use such arguments in the future.)

Thus $(AP)^{0+}_{0+}$, as an algebra, is isomorphic to direct sum of $|\mathcal{C}|$ copies of $\mathbb{C}$ and $\{E^{0^+}_C : C \in \mathcal{C}\}$ are the minimal central projections of $(AP)^{0+}_{0+}$ (clearly, $(E^{0^+}_C)^* = E^{0^+}_C$).

Case 0^-: For $g \in G$, set $E^{0^-}_g = \delta^{-1}[\psi^2_{0^-,0^-}(T^{0^-,g,0^-})]$. It is a matter of playing with the relations and isotopy of strings to establish that $E^{0^-}_g \cdot E^{0^-}_h = E^{0^-}_{gh}$, for all $g, h \in G$. Since $\{E^{0^-}_g : g \in G\}$ forms a basis by Theorem 2.11, therefore $(AP)^{0-}_{0-}$ is isomorphic to $\mathbb{C}G$ as an algebra. Also it is easy to see that $(E^{0^-}_g)^* = E^{0^-}_{g^{-1}}$. To find the irreducible $*$-representations, we need the basis of $(AP)^{0-}_{0-}$ in the matrix units form. Let $\Pi(G)$ be the set of mutually non-isomorphic irreducible unitary representations of G and let d_π be the dimension of the representation, $\pi \in \Pi(G)$. Define $E^{0^-}_{\pi ij}$ as $d_\pi \delta^{-2} \sum_{g\in G} \overline{\pi_{ij}(g)} E^{0^-}_g$ for $1 \leqslant i, j \leqslant d_\pi$. We have $\{E^{0^-}_{\pi ij} : \pi \in \Pi(G), 1 \leqslant i, j \leqslant d_\pi\}$ as matrix units for $(AP)^{0-}_{0-}$.

Case 1: First note that for each $g \in G$, there exists unique $C \in \mathcal{C}$ and unique $d \in D_C$ such that $g = d^{-1}cd$, and secondly the set $\{d^{-1}ed' : e \in E_C, d' \in D_C\}$ forms a set of representatives of distinct left cosets of $\langle g\rangle$ in G. Now, for $C \in \mathcal{C}$, $d_1, d_2 \in D_C$, $\rho \in \Pi(\langle c\rangle\backslash N_C)$ (= set of mutually non-isomorphic unitary representations of the group $\langle c\rangle\backslash N_C$; $\langle c\rangle$ is trivially a normal subgroup of N_C and so $N_C/\langle c\rangle$ is a group) and $1 \leqslant i_1, i_2 \leqslant d_\rho$ (= dimension of the representation ρ), define

$$E^{(C,\rho)}_{(d_1,i_1),(d_2,i_2)} = \frac{d_\rho\ o_C}{\delta\ n_C} \sum_{e\in E_C} \overline{\rho_{i_1 i_2}(\langle c\rangle e)}\ [\psi^2_{1,1}(T^{1,d_1{}^{-1}ed_2}_{d_1{}^{-1}cd_1,1})].$$

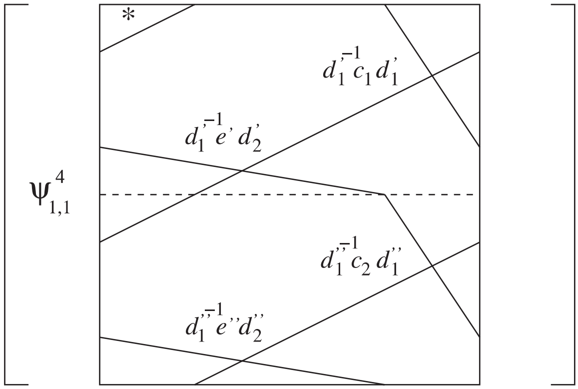
Observe that $E^{(C,\rho)}_{(d_1,i_1),(d_2,i_2)} \neq 0$, since it is a non-zero linear combination of the basis obtained in Theorem 2.11(v). We now prove that these elements form matrix units of $(AP)^1_1$.

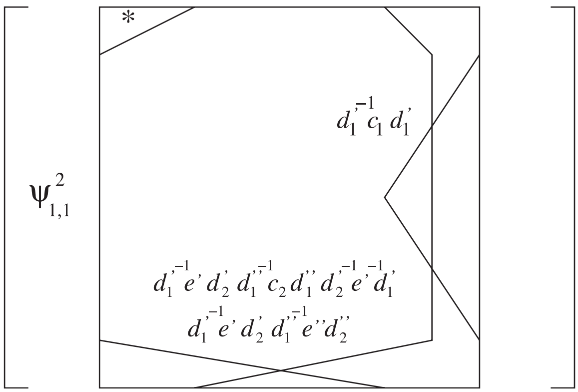
Assertions. (i) $E^{(C_1,\rho_1)}_{(d'_1,i'_1),(d'_2,i'_2)} \cdot E^{(C_2,\rho_2)}_{(d''_1,i''_1),(d''_2,i''_2)} = \delta_{(C_1,\rho_1),(C_2,\rho_2)} \delta_{(d'_2,i'_2),(d''_1,i''_1)} E^{(C_1,\rho_1)}_{(d'_1,i'_1),(d''_2,i''_2)}$.

(ii) $(E^{(C,\rho)}_{(d_1,i_1),(d_2,i_2)})^* = E^{(C,\rho)}_{(d_2,i_2),(d_1,i_1)}$.

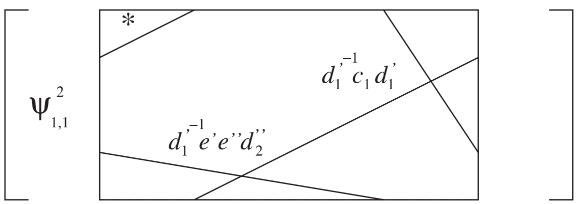
Proof of (i). For $e' \in E_{C_1}$, $e'' \in E_{C_2}$, we consider the product

$$[\psi^2_{1,1}(T^{1,d_1'^{-1}e'd'_2}_{d_1'^{-1}c_1d'_1,1})][\psi^2_{1,1}(T^{1,d_1''^{-1}e''d''_2}_{d_1''^{-1}c_2d''_1,1})]$$

$=$ 

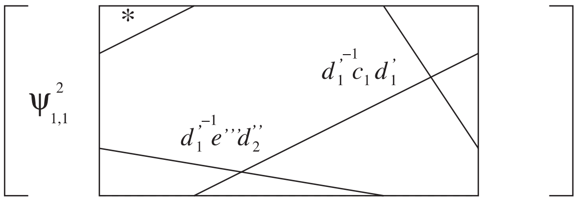
$=$ 

(by using relations of the group planar algebra and isotopy of strings)

$= \delta \; \delta_{c_1, c_2} \, \delta_{d_2', d_1''}$ 

(by using relations of the group planar algebra and the fact $d_1'^{-1} c_1 d_1' = d_1'^{-1} e' d_2' d_1''^{-1} c_1 d_1'' d_2'^{-1} e'^{-1} d_1'$ implies $C_1 = C_2$ and $d_1'' = d_2'$)

$$= \delta \ \delta_{c_1, c_2} \delta_{d_2', d_1''}$$



where $e''' \in E_{C_1}$ such that $\langle c_1\rangle e''' = \langle c_1\rangle e'e''$. Rest of the proof follows from the fact that for any group G, $\pi, \rho \in \Pi(G)$ (= set of mutually non-isomorphic irreducible unitary representations of G), and $1 \leqslant i, j \leqslant d_\pi$, $1 \leqslant k, l \leqslant d_\rho$, $g \in G$, we have

$$d_\pi \ d_\rho \ |G|^{-2} \sum_{g_1, g_2 \in G \text{ such that } g_1 g_2 = g} \overline{\pi_{i,j}(g_1) \ \rho_{k,l}(g_2)} = \delta_{\pi,\rho} \ \delta_{j,k} \ d_\pi \ |G|^{-1} \overline{\pi_{i,l}(g)}.$$

Proof of (ii).

$$\begin{aligned}
(E^{(C,\rho)}_{(d_1,i_1),(d_2,i_2)})^* &= \frac{d_\rho \ o_C}{\delta \ n_C} \sum_{e \in E_C} \rho_{i_1 i_2}(\langle c\rangle e) \ [(\psi^2_{1,1}(T^{1,d_1{}^{-1}ed_2}_{d_1{}^{-1}cd_1,1}))^*] \\
&= \frac{d_\rho \ o_C}{\delta \ n_C} \sum_{e \in E_C} \rho_{i_1 i_2}(\langle c\rangle e) \ [(\psi^2_{1,1}(T^{d_1{}^{-1}ed_2,1}_{d_1{}^{-1}cd_1,1}))] \\
&= \frac{d_\rho \ o_C}{\delta \ n_C} \sum_{e \in E_C} \overline{\rho_{i_2 i_1}(\langle c\rangle e^{-1})} \ [(\psi^2_{1,1}(T^{1,d_2{}^{-1}e^{-1}d_1}_{d_2{}^{-1}cd_2,1}))] \\
&= \frac{d_\rho \ o_C}{\delta \ n_C} \sum_{e \in E_C} \overline{\rho_{i_2 i_1}(\langle c\rangle e)} \ [(\psi^2_{1,1}(T^{1,d_2{}^{-1}ed_1}_{d_2{}^{-1}cd_2,1}))] \\
&= E^{(C,\rho)}_{(d_2,i_2),(d_1,i_1)}.
\end{aligned}$$

At the first equation above, and at numerous other places in the sequel, we use the following identity

$$[\psi^l_{m,n}(T^{\underline{x},\underline{y}}_{\underline{g},\underline{h}})^*] = [\psi^l_{m,n}(T^{\underline{y},\underline{x}}_{\underline{g},\underline{h}})],$$

which is a consequence of the definitions of these tangles and of the adjoint in the annular category. Thus, assertion (ii) is proved. Clearly the set

$$\{E^{(C,\rho)}_{(d_1,i_1),(d_2,i_2)} : C \in \mathcal{C}, \rho \in \Pi(\langle c\rangle \backslash N_C), d_1, d_2 \in D_C, 1 \leqslant i_1, i_2 \leqslant d_\rho\}$$

forms a basis for $(AP)^1_1$ since it is obtained from the basis in Theorem 2.11(v) by an invertible linear transformation given by the irreducible representations of the group

$\langle c\rangle\backslash N_C$ for $C \in \mathcal{C}$. Also from the assertion, it is apparent that the set forms a set matrix units for the algebra $(AP)_1^1$ with center indexed by $\{(C,\rho) : C \in \mathcal{C}, \rho \in \Pi(\langle c\rangle\backslash N_C)\}$ ($= I$ say). For $(C,\rho) \in I$, the summand of $(AP)_1^1$ corresponding to (C,ρ) is $M_{|D_C|}(\mathbb{C}) \otimes M_{d_\rho}(\mathbb{C})$. Hence,

$$(AP)_1^1 \cong \bigoplus_{C\in\mathcal{C}} \bigoplus_{\rho\in\Pi(\langle c\rangle\backslash N_C)} (M_{|D_C|}(\mathbb{C}) \otimes M_{d_\rho}(\mathbb{C})). \qquad \square$$

Theorem 2.15. (i) *There are exactly* $|\mathcal{C}|$ *many mutually non-isomorphic irreducible* $*$*-representations with weight* 0^+.

(ii) *There are exactly* $|\Pi(G)|$ *many mutually non-isomorphic irreducible* $*$*-representations with weight* 0^-.

Proof. (i) Fix $C \in \mathcal{C}$. Define a representation, $F^{0^+,C}$ by the following rule:

$$F^{0^+,C}(n) = (AP)_{0^+}^n \cdot E_C^{0^+}$$

for $n \in Col$. More explicitly, we have the following (by using Theorem 2.11, relations of the group planar algebra and isotopy of strings):

$$F^{0^+,C}(n) = span\{[\psi^2_{n,0^+}(T^{\underline{x},0^+}_{d^{-1}cd,1})] : d \in D_C, \underline{x} \in G^n \text{ such that } x_1 = 1\},$$

for $n \in \mathbb{N}$,

$$F^{0^+,C}(0^-) = (AP)_{0^+}^{0^-} \cdot E_C^{0^+} = \begin{cases} 0 & \text{if } C \neq \{1\}, \\ \mathbb{C}[A_{0^+}^{0^-}] & \text{otherwise} \end{cases}$$

and $F^{0^+,C}(0^+) = \mathbb{C}E_C^{0^+}$. By definition, $\{F^{0^+,C}(n)\}_{n\in Col}$ is closed under composition by elements of $(AP)_n^m$, that is, $a \cdot v \in F^{0^+,C}(m)$, for all $v \in F^{0^+,C}(n)$, $a \in (AP)_n^m$. For $a \in (AP)_n^m$, define $F^{0^+,C}(a)$ to be the linear map induced by composition. This makes $F^{0^+,C}$ a representation.

To make $F^{0^+,C}$ into a $*$-representation, we define an inner product. Let τ be a faithful state on $(AP)_{0^+}^{0^+}$ (which also becomes a trace since the algebra is abelian). Define

$$\langle v, w\rangle = \tau(w^* v)$$

for $v, w \in F^{0^+,C}(n)$, $n \in Col$. We need to prove that $\langle\cdot,\cdot\rangle$ is a positive definite. We prove this for $F^{0^+,C}(n)$ for $n \in \mathbb{N}$—other cases are easy. Let us take any non-zero element of $F^{0^+,C}(n)$,

$$v = \sum_{\underline{x}\in G^n \text{ such that } x_1=1,\ d\in D_C} \lambda_{\underline{x},d}\ [\psi^2_{n,0^+}(T^{\underline{x},0^+}_{d^{-1}cd,1})] \neq 0.$$

Using the relations of group planar algebra and isotopy of strings, we can deduce the following equation for $\underline{x}, \underline{y} \in G^n$, such that $x_1 = 1 = y_1$, $d_1, d_2 \in D_C$:

$$[\psi^2_{n,0^+}(T^{\underline{x},0^+}_{d_1^{-1}cd_1,1})]^* \cdot [\psi^2_{n,0^+}(T^{\underline{y},0^+}_{d_2^{-1}cd_2,1})] = \delta_{\underline{x},\underline{y}}\ \delta_{d_1,d_2}\ \delta^{(n-1)}\ n_C\ E^{0^+,C}$$

(where $x_1 = 1 = y_1$ plays a crucial role). Thus, we have

$$\langle v, v\rangle = \delta^{2(n-1)}\ n_C^2 \sum_{\underline{x}\in G^n \text{ such that } x_1=1,\ d\in D_C} |\lambda_{\underline{x},d}|^2\ \tau(E^{0^+,C}) > 0.$$

From the definition of the inner product, it easily follows that $\langle av, w\rangle = \langle v, a^*w\rangle$ for $a \in (AP)^m_n$, $v \in F^{0^+,C}(n)$, $w \in F^{0^+,C}(m)$, for all $m, n \in Col$—thus $F^{0^+,C}$ is $*$-preserving. Irreducibility of $F^{0^+,C}$ follows from that of $F^{0^+,C}(0^+)$ as a module over the algebra $(AP)^{0^+}_{0^+}$. Moreover, if $C_1 \neq C_2$, then F^{0^+,C_1} is not isomorphic to F^{0^+,C_2}, since E^{0^+,C_1} kills $F^{0^+,C_2}(0^+)$ whereas acts as identity on the corresponding object for C_1.

(ii) Fix $\pi \in \Pi(G)$. Define a representation, $F^{0^-,\pi}$ in the following way:

$$F^{0^-,\pi}(n) = span\left\{ \bigcup_{1\leqslant i\leqslant d_\pi} (AP)^n_{0^-}\ E^{0^-}_{\pi i 1} \right\}$$

for $n \in Col$. Moreover, it is quite straightforward to find $F^{0^-,\pi}(0^\pm)$ explicitly, by using Theorem 2.11, relations of the group planar algebra and isotopy of strings:

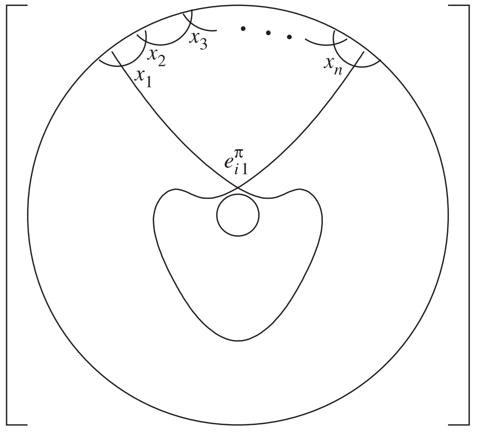
$$F^{0^-,\pi}(0^+) = \begin{cases} \mathbb{C}[A^{0^+}_{0^-}] & \text{if } \pi \text{ is the trivial representation,} \\ \{0\} & \text{otherwise} \end{cases}$$

and $F^{0^-,\pi}(0^-) = span\{E^{0^-}_{\pi i 1} : 1 \leqslant i \leqslant d_\pi\}$. By definition, $\{F^{0^-,\pi}(n)\}_{n\in Col}$ is closed under composition by elements of $(AP)^m_n$, that is, $a \cdot v \in F^{0^-,\pi}(m)$, for all $v \in F^{0^-,\pi}(n)$, $a \in (AP)^m_n$. For $a \in (AP)^m_n$, define $F^{0^-,\pi}(a)$ to be the linear map induced by composition. This makes $F^{0^-,\pi}$ a representation.

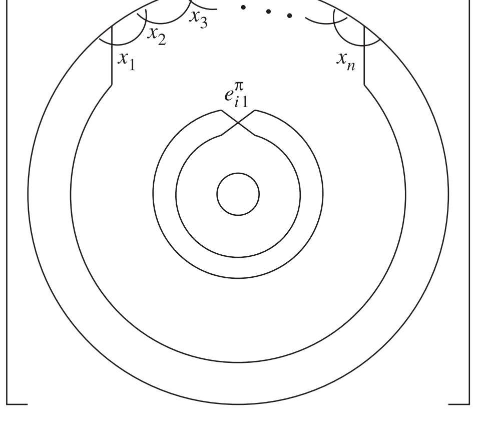
Let τ be the trace on $(AP)^{0^-}_{0^-}$ coming from the one on $\mathbb{C}G$ which is given by the indicator function of the identity of the group. Define

$$\langle v, w\rangle = \tau(w^* v)$$

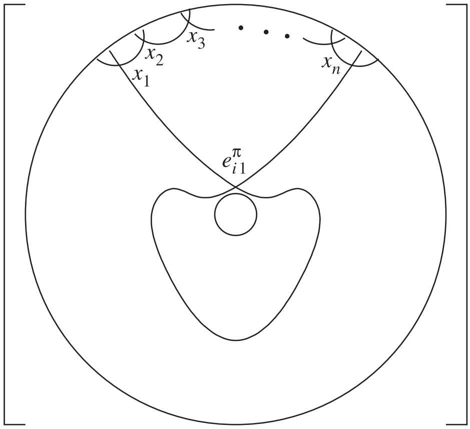
for $v, w \in F^{0^-,\pi}(n)$, $n \in Col$. Again, we will prove positive definiteness for $F^{0^-,\pi}(n)$ for $n \in \mathbb{N}$—other cases are easy. For this, we use the fact that $F^{0^-,\pi}(n)$ is spanned by elements of the form:



($= E_{\underline{x},i}$ say) for $\underline{x} \in G^n$ such that $x_1 = 1$, $1 \leqslant i \leqslant d_\pi$ where $e^{\pi}_{ij} = d_\pi |G|^{-1} \sum_{g\in G} \overline{\pi_{ij}(g)} g$. To see this, we pick an element in the basis of $(AP)^n_{0^-}$ obtained in Theorem 2.11(iii), namely $[\psi^1_{n,0^-}(T^{\underline{x},0^-})]$, and compose it with $E^{0^-}_{\pi i 1}$. The composition will be a scalar multiple of the following picture:

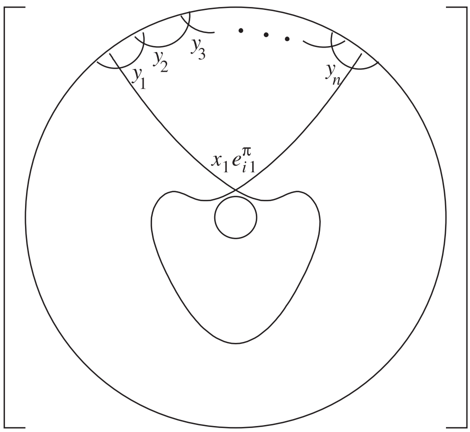


$= \delta$



(using relations of group planar algebra)

$= \delta$



(using relations of group planar algebra) where $\underline{y} = x_1^{-1} \cdot \underline{x}$. Note that $x_1 e^{\pi}_{i1}$ is again a linear combination of e^{π}_{j1} for $1 \leqslant j \leqslant d_\pi$ and also $y_1 = 1$. Each of the three diagrams above is to be interpreted as the appropriate linear combination of terms with e^{π}_{i1} replaced by g's. We shall adopt this convention in the sequel without further explanation.

Now using relations in group planar algebra and isotopy of strings, we can deduce:

$$(E_{\underline{x},i})^* \ E_{\underline{y},j} = \delta_{(\underline{x},i),(\underline{y},j)} \ \delta^n E^{0^-}_{\pi 11}$$

for $\underline{x}, \underline{y} \in G^n$ such that $x_1 = 1 = y_1$ (which plays a vital role) and $1 \leqslant i, j \leqslant d_\pi$. Since $\tau(E^{0^-}_{\pi 11}) > 0$, therefore, using the above identity, we have $\langle v, v \rangle > 0$ for all non-zero v in $span\{E_{\underline{x},i} : \underline{x} \in G^n$ such that $x_1 = 1, 1 \leqslant i \leqslant d_\pi\} = F^{0^-,\pi}(n)$. It trivially follows from the definition of the inner product that $\langle av, w \rangle = \langle v, a^* w \rangle$ for $a \in (AP)^m_n$,

$v \in F^{0^-,\pi}(n)$, $w \in F^{0^-,\pi}(m)$, for all $m, n \in Col$. Thus $F^{0^-,\pi}$ is a $*$-representation. Irreducibility of $F^{0^-,\pi}$ follows from that of $F^{0^-,\pi}(0^-)$ as a module over the algebra $(AP)^{0^-}_{0^-}$.

We can apply the same reasoning as in (i), to show that $*$-representations coming from distinct elements of $\Pi(G)$ are non-isomorphic. $\square$

Remark 2.16. The $*$-representation in (i) (resp. (ii)) corresponding to $C = \{1\}$ (resp. trivial representation of G) is isomorphic to the trivial representation of P.

Remark 2.17. In the proof of Theorem 2.15, we describe an explicit spanning set of the irreducible $*$-representations and this spanning set is orthogonal with respect to the inner product defined. Enumerating the number of elements of the orthogonal spanning set, we get:

(i) for $C \in \mathcal{C}$, $dim(F^{0^+,C}(n)) = |C|.|G|^{n-1}$ for $n \in \mathbb{N}$, $dim(F^{0^+,C}(0^+)) = 1$ and $dim(F^{0^+,C}(0^-)) = \delta_{C,\{1\}}$, and the dimension of $F^{0^+,C}$ is given by

$$\Phi_{F^{0^+,C}}(z) = \frac{1}{2} + \frac{\delta_{C,\{1\}}}{2} + \frac{|C|z}{1-|G|z}$$

with radius of convergence $\frac{1}{|G|}$,

(ii) for $\pi \in \Pi(G)$, $dim(F^{0^-,\pi}(n)) = d_\pi.|G|^{n-1}$ for $n \in \mathbb{N}$, $dim(F^{0^-,\pi}(0^+)) = \delta_{\pi,triv}$ and $dim(F^{0^-,\pi}(0^-)) = d_\pi$, and the dimension of $F^{0^-,\pi}$ is given by

$$\Phi_{F^{0^-,\pi}}(z) = \frac{\delta_{\pi,triv}}{2} + \frac{d_\pi}{2} + \frac{d_\pi z}{1-|G|z}$$

with radius of convergence $\frac{1}{|G|}$.

To find the irreducible $*$-representations of weight 1, we first consider the ideal $\widehat{(AP)^1_1}$ which is the span of union of the following two ideals:

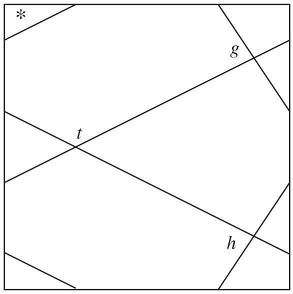
$$span\{a \cdot b : a \in (AP)^1_{0^+},\ b \in (AP)^{0^+}_1\} = R_+,$$

$$span\{a \cdot b : a \in (AP)^1_{0^-},\ b \in (AP)^{0^-}_1\} = R_-.$$

To find the irreducible summands in $(AP)^1_1$ which are also in R_+, we pick any elements in the basis of $(AP)^1_{0^+}$ and $(AP)^{0^+}_1$ obtained in Theorem 2.11 and compose them.

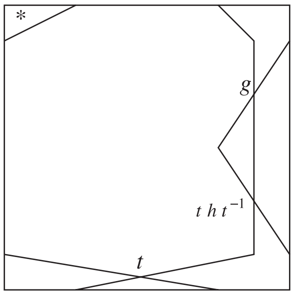
Typically we have $[\psi^2_{1,0^+}(T^{1,0^+}_{g,1})] \cdot [\psi^2_{0^+,1}(T^{0^+,1}_{h,1})]$

$$= \delta^{-1} \sum_{t \text{ in } G} \Psi^4_{1,1}$$



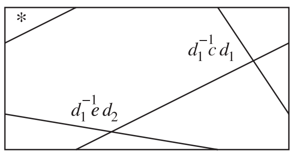
(using relations of the group planar algebra)

$$= \delta^{-1} \sum_{t \text{ in } G} \Psi^2_{1,1}$$



(using relations of the group planar algebra and isotopy)

$$= o_C \sum_{e \text{ in } E_C} \Psi^2_{1,1}$$



(using Lemma 2.1 (b) and assuming g and h are conjugate otherwise it becomes zero; in particular, $g = d_1^{-1}cd_1$, $h = d_2^{-1}cd_2$ for $d_1, d_2 \in D_C$ which implies t should take values in $d_1^{-1}N_Cd_2 = \amalg_{e \in E_C} d_1^{-1}\langle c\rangle ed_2$ in order for the final result to be non-zero). Looking at the generators of R_+ carefully, we can conclude that the irreducible summands of $(AP)^1_1$ contained in the ideal, R_+ correspond to $\{(C, \rho) : C \in \mathcal{C}, \rho = triv \in \Pi(\langle c\rangle \backslash N_C)\}$.

Similarly, for R_-, we compose two elements from the basis of $(AP)^1_{0^-}$ and $(AP)^{0^-}_1$ obtained in Theorem 2.11. Typically, we have

$$[\psi^1_{1,0^-}(T^{g,0^-})] \cdot [\psi^1_{0^-,1}(T^{0^-,h})] \;=\; [\psi^2_{1,1}(T^{g,h}_{1,1})] \;=\; [\psi^2_{1,1}(T^{1,g^{-1}h}_{1,1})]$$

for all $g, h \in G$. In view of the invertibility of the matrix $((\rho_{ij}(g)))$ (with rows and columns indexed, respectively, by the (ρ, i, j)'s and the g's), we may now deduce that the irreducible summands of $(AP)^1_1$ contained in the ideal, R_- correspond to $\{(\{1\}, \rho) : \rho \in \Pi(G)\}$.

Let $\tilde{I}$ be $\{(C, \rho) \in I : C \neq \{1\}, \rho \neq triv \in \Pi(\langle c\rangle \backslash N_C)\}$. It is clear that the irreducible representations of $(LWP)_1$ is indexed by $\tilde{I}$.

Theorem 2.18. *The set of isomorphism classes of irreducible $*$-representations of weight* 1 *of the group planar algebra is indexed by $\tilde{I}$.*

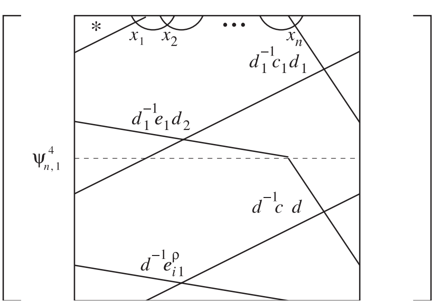
Proof. Fix $(C, \rho) \in \tilde{I}$. Define a representation, $F^{C,\rho}$ in the following way:

$$F^{C,\rho}(n) = \ span\left\{ \bigcup_{d\in D_C, 1\leqslant i\leqslant d_\rho} (AP)^n_1 \cdot E^{(C,\rho)}_{(d,i),(1_G,1)} \right\}.$$

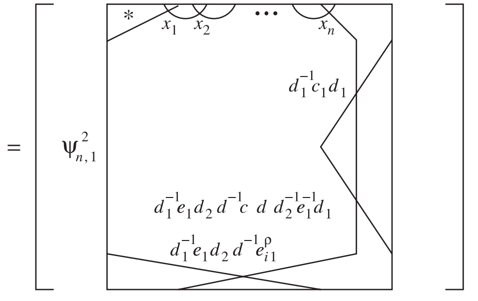
Since any element of $(AP)^{0\pm}_1$ can be expressed as a composition of elements from $(AP)^{0\pm}_1$ and $\widehat{(AP)}{}^1_1$ and $(C, \rho) \in \tilde{I}$, therefore $F^{C,\rho}(0^\pm) = \{0\}$. Also it is fairly clear from the algebraic structure of $(AP)^m_n$ that $F^{C,\rho}(1) = \ span\{\bigcup_{d\in D_C,1\leqslant i\leqslant d_\rho} E^{(C,\rho)}_{(d,i),(1_G,1)}\}$. By the definition of $F^{C,\rho}$, $\{F^{C,\rho}(n)\}_{n\in Col}$ is closed under composition of elements of $(AP)^m_n$, that is, $a \cdot v \in F^{C,\rho}(m)$ for all $a \in (AP)^m_n$, $v \in F^{C,\rho}(n)$. For $a \in (AP)^m_n$, define $F^{C,\rho}(a)$ as the linear map induced from composition by a. Thus $F^{C,\rho}$ forms a representation with weight greater than 0.

To make this into a $*$-representation, we choose a faithful trace τ on $(AP)^1_1$. Define $\langle v, w\rangle = \tau(w^*v)$ for all $v, w \in F^{C,\rho}(n)$, $n \in Col$. It is easy to see that $\langle av, w\rangle = \langle v, a^*w\rangle$ for $a \in (AP)^m_n$, $v \in F^{C,\rho}(n)$, $w \in F^{C,\rho}(m)$, for all $m, n \in Col$. Assuming the positive definiteness of $\langle\cdot,\cdot\rangle$, $F^{C,\rho}$ is clearly $*$-preserving and irreducibility will also follow because $span\{\bigcup_{d\in D_C,1\leqslant i\leqslant d_\rho} E^{(C,\rho)}_{(d,i),(1_G,1)}\}$ is irreducible as a module over the algebra $(AP)^1_1$.

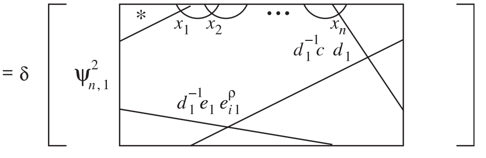
To prove positive definiteness, first we find a 'good' spanning set of $F^{C,\rho}(n)$ for $n \in \mathbb{N}$. Any element in the basis of $(AP)^n_1$ obtained in Theorem 2.11 will be of the form $[\psi^2_{n,1}(T^{\underline{x},d_1^{-1}e_1d_2}_{d_1^{-1}c_1d_1,1})]$ for $\underline{x} \in G^n$ such that $x_1 = 1$, $C_1\ \mathcal{C}$, $d_1, d_2 \in D_{C_1}$, $e_1 \in E_{C_1}$. The composition $[\psi^2_{n,1}(T^{\underline{x},d_1^{-1}e_1d_2}_{d_1^{-1}c_1d_1,1})] \cdot E^{(C,\rho)}_{(d,i),(1_G,1)}$ is equal to a scalar times the following picture:



($e^{\rho}_{i,j} = d_\rho \ o_C \ N_C^{-1} \sum_{e \in E_C} \overline{\rho_{ij}(\langle c \rangle e)} \ e$, for $1 \leqslant i, j \leqslant d_\rho$)



(using relations of group planar algebra and isotopy of strings)



(using relations of group planar algebra and assuming $C = C_1$ and $d_2 = d$ otherwise it is zero since $d_1^{-1} e_1 d_2 d^{-1} c d d_2^{-1} e_1 d_1 = d_1^{-1} c_1 d_1$ implies $C = C_1$ and $d_2 = d$). Now $e_1 e^{\rho}_{i1}$ can be expressed as a linear combination of terms of the form $c^k e^{\rho}_{j1}$ ($1 \leqslant k \leqslant o_C$, $1 \leqslant j \leqslant d_\rho$). An appeal to Lemma 2.1(b) now shows that $F^{C,\rho}(n)$ is spanned by the set

$$\{[\psi^2_{n,1}(T^{\underline{x},\ d^{-1} e^{\rho}_{i1}}_{d^{-1} c d, 1})] : \underline{x} \in G^n \text{ such that } x_1 = 1, d \in D_C, 1 \leqslant i \leqslant d_\rho\}.$$

Now, we consider the composition

$$[\psi^2_{n,1}(T^{\underline{x},\ d_1^{-1} e^{\rho}_{i1}}_{d_1^{-1} c d_1, 1})]^* \cdot [\psi^2_{n,1}(T^{\underline{y},\ d_2^{-1} e^{\rho}_{j1}}_{d_2^{-1} c d_2, 1})]$$

(for $\underline{x}, \underline{y} \in G^n$ such that $x_1 = 1 = y_1$, $d_1, d_2 \in D_C$, $1 \leqslant i, j \leqslant d_\rho$)

$$= \frac{d_\rho \ o_C}{N_C} \sum_{e \in E_C} \rho_{i1}(\langle c \rangle e) \ [\psi^2_{1,n}(T^{d_1^{-1} e,\ \underline{x}}_{d_1^{-1} c d_1, 1})] \cdot [\psi^2_{n,1}(T^{\underline{y},\ d_2^{-1} e^{\rho}_{j1}}_{d_2^{-1} c d_2, 1})]$$

$$= \delta_{\underline{x},\underline{y}} \ \frac{d_\rho \ o_C \ \delta^{n-1}}{N_C} \sum_{e \in E_C} \rho_{i1}(\langle c \rangle e) \ [\psi^4_{1,1}(T^{d_1^{-1} e,\ d_2^{-1} e^{\rho}_{j1}}_{(d_1^{-1} c d_1, d_2^{-1} c d_2), (1,1)})]$$

(using relations of group planar algebra and $x_1 = 1 = y_1$)

$$= \delta_{d_1,d_2}\ \delta_{\underline{x},\underline{y}}\ \frac{d_\rho\ o_C\ \delta^{n-1}}{N_C} \sum_{e \in E_C} \rho_{i1}(\langle c\rangle e)\ [\psi^2_{1,1}(T_{c,1}^{1,e^{-1}e^{\rho}_{j1}})]$$

(using relations of group planar algebra and isotopy of strings)

$$= \delta_{d_1,d_2}\ \delta_{\underline{x},\underline{y}}\ \frac{d_\rho\ o_C\ \delta^{n-1}}{N_C} \sum_{e \in E_C} \overline{\rho_{1i}(\langle c\rangle e^{-1})}\ [\psi^2_{1,1}(T_{c,1}^{1,e^{-1}e^{\rho}_{j1}})]$$

$$= \delta_{d_1,d_2}\ \delta_{\underline{x},\underline{y}}\ \delta^{n-1}\ [\psi^2_{1,1}(T_{c,1}^{1,e^{\rho}_{1i}e^{\rho}_{j1}})]$$

$$= \delta_{i,j}\ \delta_{d_1,d_2}\ \delta_{\underline{x},\underline{y}}\ \delta^{n-1}\ [\psi^2_{1,1}(T_{c,1}^{1,e^{\rho}_{11}})]$$

which has positive τ-value, if $i = j$, $d_1 = d_2$, $\underline{x} = \underline{y}$. Using these facts, it trivially follows that $\langle\cdot,\cdot\rangle$ is positive definite.

To show that the $*$-representations coming from two distinct elements, (C_i, ρ_i), $i = 1, 2$, in $\tilde{I}$ are non-isomorphic, note that the minimal central projection in $(AP)^1_1$ corresponding to (C_1, ρ_1) kills the second representation completely, at grade 1 and acts as identity on the other. $\square$

Remark 2.19. In the proof of Theorem 2.18, we describe an explicit spanning set of the irreducible $*$-representations and this spanning set is orthogonal with respect to the inner product defined. Enumerating the number of element of the orthogonal spanning set, we obtain, for $(C, \rho) \in \tilde{I}$, $dim(F^{C,\rho}(n)) = d_\rho.|C|.|G|^{n-1}$ for $n \in \mathbb{N}$, and the dimension of $F^{0^+,C}$ is given by

$$\Phi_{F^{C,\rho}}(z) = \frac{d_\rho |C| z}{1 - |G| z}$$

with radius of convergence $\frac{1}{|G|}$.

Hence, we assert the validity of an affirmative answer to Question 1.12 for the case of group planar algebras.

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